Cardinal Utility and Contrast in Models of Choice and Response Time

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Abstract

Response times can help reveal preferences when choices are noisy. This paper introduces a novel framework to analyze models that predict choices and response times from the utility of the alternatives. Decomposing models based on which choice and response time distributions they generate and how these distributions are normatively interpreted, I uncover a key elasticity - the contrast level - that drives their behavioral predictions. However, the contrast level is arbitrarily set and not identifiable by current calibration methods. I propose a new calibration method that I illustrate on practical examples with data simulated from a Drift Diffusion Model. My results shed light on the normative implications of a wide class of models from cognitive science and offer guidance on their use in economics. They are relevant wherever random utility models are currently used, for instance demand models or discrete choice experiments.

1 Introduction

There is an increasing interest in the use of response times in the analysis of economic decisions. Response times, on their own or combined with choice data, have been shown to predict preferences in various contexts [Xiang Chiong et al., 2023, Berlinghieri et al., 2023, Alós-Ferrer et al., 2021, Alos-Ferrer and Garagnani, 2023, Li and Bansal, 2024]. This is particularly relevant for revealing preferences, as measured by cardinal utility, when we expect decision-makers to make noisy or biased choices.

Most of the aforementioned studies rely on structural models from cognitive science that link preferences over alternatives to choices and response times. These models - often called evidence accumulation models - describe decision-making as a noisy process of information acquisition over time that continues until a threshold level of evidence is reached and determines a choice. In these models, preferences are usually assumed to determine the rate of the evidence accumulation in favor of the alternatives..

There are plenty of such models in the cognitive science literature, but very few are used in economics. Prominent examples of evidence accumulation models are the Drift-Diffusion Model (DDM) [Ratcliff and McKoon, 2008, Ratcliff et al., 2016] and the Linear Ballistic Accumulator (LBA) model [Brown and Heathcote, 2008a]. So far, economists have almost exclusively focused on the DDM model and its variants. The so-called "horse-race models" such as the LBA have received little attention past the seminal article from Marley and Colonius [1992], even though they are conceptually very similar to the popular logit and probit models and are directly applicable to non-binary choices.

Cognitive scientists themselves are uncertain about how these models differ from each other. From an empirical perspective, the most widespread models have been shown to fit comparably well the observed choice and response time distributions in a wide range of tasks [Donkin et al., 2011]. From a theoretical perspective, it has been argued [Jones and Dzhafarov, 2014] that these models impose almost no restriction on the output distribution, in the sense that differences between them in that respect are only driven by the specific parametric assumptions in their noise terms.

Understanding which differences between models matter for measuring the utility of the alternatives is crucial. It is an essential prerequisite to broadening the range of models of choice and response time that can safely be used to address normative questions in economics. Otherwise, applied economic research will remain largely confined to a handful of DDM-inspired models, bound to the study of binary choices and ignorant of the thorough empirical work in mathematical psychology based on alternative models. However, we lack concepts and a shared language to analyze how the cardinal utility of each alternative affects the output distribution of choice and response time in a model.

In this paper, I propose a theoretical framework to compare models of choice and response times, focusing on their use for the measuring utility. A model is understood as a function that maps the utility of a fixed number of alternatives to an output distribution of choice and response time. Building on an equivalence result from Marley and Colonius [1992] related to competing risk models in survival analysis, I show that most models in the literature can be encapsulated in two functions: a "range" and a "contrast". The "range" is a family of cumulative distribution functions indexed by a scalar parameter. Each function describes the latent distribution of response times for a choice alternative. The "contrast" is a real-valued function that determines the parameter of the range for one alternative from the utility of all the alternatives. Finally, the choice is the outcome of a horse-race between the alternative-specific response times. When revealing cardinal preferences from choice and response time, the range and the contrast play a complementary role: the range decodes the observed distribution of choice and response times and attributes to each alternative a real number, while the contrast interprets these numbers in terms of utility. The conditions under which a model can be described by its range and contrast are extremely weak.

Equipped with these novel concepts of range and contrast, this paper is able to formulate and address the following questions.

- What is the set of models for which such a range-contrast representation exist? In what sense do the range and contrast characterize a model? Given a numerical description of two models, how can I compute and compare their range and their contrast? (Section 3)
- How do the properties of the contrast relate to the behavioral predictions of the model? Which restrictions on the contrast have been commonly imposed in the literature? What is specific about the contrast of the DDM? (Section 4)
- What data do I need to identify the range and the contrast of a model? What do I need to know about the model to reveal cardinal preferences from choice and response time data? What are the differences between the LBA and DDM model in terms of range and contrast? How to calibrate the contrast in practice? Which models can be used for revealing cardinal preferences? (Section 5)

One important contribution of this paper is the notion of contrast level introduced in Section 4. The contrast level is a key elasticity of the contrast function that plays a key role in my answer to the previous questions. It measures the relative effect on the speed of latent response times for an alternative of an increase in the utility of another alternative as compared to an increase in its own utility. In existing models of the literature, the contrast level is constant and takes only two values. It is equal to -1 in DDM-like models, which means that increasing the utility of an alternative has the same effect as decreasing the utility of another alternative by the same amount, and 0 in race models like the LBA, which means that the latent response times associated to an alternative are not affected by the utility of another. These two values for the contrast level generate diverging behavioral predictions, insofar as the utility of the alternatives vary in a known amount. However, they may still generate exactly the same output distribution of choice and response times - just for different input utilities. For this reason, the contrast level is typically not identified by the standard practices for model calibration in cognitive science, in which the drift rate is a free parameter of the model and variations in utility are unobserved.

This paper does not make any empirical claim about what is the correct value of the contrast level. Instead, it establishes the theoretical and econometric foundations for the use of models with an arbitrary constant contrast level and the calibration of this parameter. I show that the contrast level has clear behavioral implications. It is also invariant under increasing transformations of the utility scale. It can be identified from changes in the joint distribution of choice and response time when some variations in utility are observed, for instance changes in price for a given good. Additionally, I show how to calibrate the contrast level in practice by introducing a flexible parameter in an existing race model.

I illustrate my theoretical framework and my calibration method by comparing the DDM and LBA models. The literature had observed that both models fit comparably well empirical distributions of choice and response times [Donkin et al., 2011]. This suggests, in my framework, that they have a comparable range. Making use of my results on normalized range-contrast representation, I can plot on the same 2D map the range of the LBA and DDM models, which provides a convenient way to inspect the differences in the distributions of choice and response time that the two model can generate. Since the range of the LBA and DDM are similar, they differ mostly by their contrast level. I calibrate a LBA with a flexible parameter for the contrast level on data generated by a DDM - making use of the fact that I know the utility of the alternatives in my simulation. Logically, the proposed estimation procedure finds a contrast level close from -1, the theoretical contrast level of the DDM. I can then investigate the amount of data required to calibrate the contrast level. The proposed theoretical framework and set of econometric methods applies equally to many variations of the DDM model - such as having a random starting point, collapsing boundaries [Fudenberg et al., 2015] or replacing the Brownian motion by a Levy flight process [Wieschen et al., 2020] - as well as any independent race models - such as the LBA [Brown and Heathcote, 2008b], the racing diffusion model [Tillman et al., 2020] or the race Lévy flight model [Rasanan et al., 2022].

My theoretical results bring a fresh perspective to the empirical literature evaluating models of choice and response times. The common practice in cognitive science is to evaluate models by their goodness-of-fit using choice and response time only. My results show that, without incorporating information about the stimulus - the utility of the alternatives -, this approach only evaluates the range of the model but not the contrast. Unless one has a deep conviction that the only reasonable contrast level is -1, this is a problem for economists. It means that current approaches to model evaluation in cognitive science and the associated literature cannot guide us in the choice of a suitable contrast level.

This suggests a natural division of labor between cognitive scientists and economists. The former can keep improving the range of the models using their usual metrics and datasets. The data requirements can be important at this stage. The latter can start from these models to have a correct range and then calibrate the contrast level using datasets that also contain information about the variations in utility, such as experiments with induced values, discrete choice experiments or choice sets with varying prices. My econometric results and

Monte Carlo calibration exercises are oriented towards these applications. This type of data is more costly to obtain because it requires a controlled environment, but the calibration of the contrast level is much less data-intensive than that of the range. By analogy with deep learning, the range can be seen as a foundational model that extract relevant features from the distribution of choice and response times, whereas the calibration of the contrast level corresponds to a fine-tuning for the specific value-based task considered. Overall, the distinction between range and contrast reveals the complementary between mindful and mindless economics [Camerer, 2008, Gul and Pesendorfer, 2008], the former contributing to the accuracy of behavioral models and the latter to the normative interpretation of these behaviors.

This paper also contributes to the theoretical literature at the intersection of economics and cognitive science investigating the properties of models of choice and response times. One strand of this literature, closer to decision-theory, has offered some axiomatization of the DDM [Fudenberg et al., 2020, Baldassi et al., 2020] and related it with classical random utility models [Fudenberg et al., 2015, 2018, Cerreia-Vioglio et al., 2023]. Alós-Ferrer et al. [2021] introduce a general framework to reveal ordinal preferences from choice and response times that shares some similarity with my paper. Both approaches apply to the case of strongly Fechnerian models, but generalize in different directions. Alós-Ferrer et al. [2021] relax this assumption by allowing the noise distribution to vary across pairs of alternatives. I relax this assumption by giving an independent role to the utility of each alternative. [1]. In the same vein, Webb [2019] and Webb et al. [2019] have tried to extend random utility models by incorporating any sort of neural correlates as predictor, including response times. So far, this literature has paid limited attention to the distinction between predicting behavior and interpreting behavior as reflecting certain preferences - as given by the notions of range and contrast - and thus has not discussed the limits to what can be calibrated depending on data availability.

Another strand of the literature, closer to mathematical psychology, had begun to unify DDM-like models and race models under a common analytical framework [Marley and Colonius, 1992, Jones and Dzhafarov, 2014] and studied the horse-race counterpart of the DDM [Ratcliff and Smith, 2004, Donkin et al., 2011, Terry et al., 2015, Khodadadi and Townsend, 2015]. In this literature, my paper puts a new perspective on a critical point raised by Jones and Dzhafarov [2014]. These authors pointed out that the models in the literature are so flexible that relaxing any of their distributional assumptions is sufficient to fit any distribution of choice and response time. In my framework, it is clear that relaxing these

[1] Note that Alós-Ferrer et al. [2021] assume a contrast level of -1 and focus on binary choice. I show in this paper that - in a certain sense - it is impossible to extend the Fechnerian version of their framework to more than two alternatives.

distributional assumptions only affects the range, not the contrast. Therefore, even if all models have the same range up to a distributional assumption, they may still have a different contrast and yield diverging behavioral predictions about the effect of varying the utility of the alternatives. My results suggest that we should accept the apparent arbitrariness of the range, because the distributional assumptions in current models have been validated in numerous empirical studies, but pay more attention to the arbitrariness of the contrast level.

The rest of the paper is structured as follows. Section 2 showcases the theoretical results through illustrative examples. In Section 3, I introduce my theoretical framework based on the range-contrast representation of a model and characterize the class of models to which I applies. In Section 4, I define the contrast level of a model and analyze its behavioral implications. In Section 5, I state and illustrate the econometrics implications of the two previous sections in relation to economic applications. I summarize the main results and discuss the possible extensions in Section 6. Each of Section 3, 4 and 5 opens with a comprehensive summary, which time-pressed readers can use as an index - or an alternative - to the formal statement of the results.

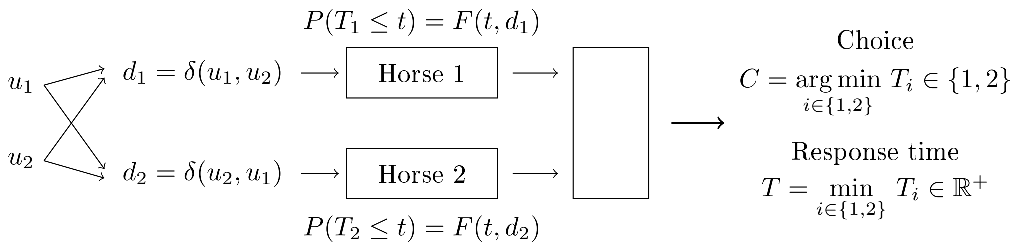
2 Illustration of the results through examples

This section provides a quick overview of the main ideas of each section through examples, using relaxed notations and focusing on the case of $n = 2$ alternatives.

Section 3 introduces the range-contrast representation of a model. The **range** is a function $F : \mathbb{R}^{+*} \times \mathbb{R} \to [0, 1]$ increasing in both arguments. The range should be understood as a distribution function, taking as argument the time t and a parameter d - readers familiar with evidence accumulation models can think of d as a drift rate in a race model. For instance, $F^{\mathrm{A}}(t, d) = 1 - \exp(-e^{d}t)$, $F^{\mathrm{B}}(t, d) = 1 - \exp(-e^{\sqrt[3]{d}}t)$ or $F^{\mathrm{C}}(t, d) = G(t + e^{d}t^2)$ for any increasing cumulative distribution function $G : \mathbb{R} :\to \mathbb{R}$.

The **contrast** is a smooth function $\delta : \mathbb{R}^n \to \mathbb{R}$ increasing in its first argument. The contrast function should be understood as generating a parameter d for the range function from the utility u_i of the alternatives. For instance, $\delta^{\mathrm{A}}(u_1, u_2) = u_1 - 2u_2$, $\delta^{\mathrm{B}}(u_1, u_2) = (u_1 - 2u_2)^3$ or $\delta^{\mathrm{C}}(u_1, u_2) = 1 - 1/\exp(u_1+\arctan(u_2))$.

The **distribution of choice and response time** (C, T) is determined from the range F and the contrast δ by a horse race. There are n running horses, one per alternative. The chosen alternative C is the one whose horse arrives first and the response time T is the arrival time T_i of this horse. The T_i are drawn independently such that $P(T_1 \leq t) = F(t, \delta(u_1, u_2))$ and $P(T_2 \leq t) = F(t, \delta(u_2, u_1))$ This decision process is summarized on Figure 1. Those



Notes: The joint distribution of choice and response time results from a horse race, symbolized by the white rectangle. The horses have random independent arrival times T_i. The cumulative distribution function of T_i is $t \mapsto F(t, d_i)$, where F is the range and d_i depends on the contrast and the utility of the alternatives.

Figure 1: Range-contrast representation of a model with $n = 2$ alternatives

familiar with accumulation models can see it as a race model in which the drift rates depend flexibly on the utility of the alternatives through the contrast. Modeling the decision process as a race between several independent horses may seem quite restrictive. Actually, I show that the conditions on the model for a such a representation to exist are extremely weak.

The **range contrast representation** (F, δ) of a model is not unique. In the previous examples of range and contrast, it is clear that the pairs $(F^{\mathrm{A}}, \delta^{\mathrm{A}})$ and $(F^{\mathrm{B}}, \delta^{\mathrm{B}})$ lead to the same distribution of choice and response time, whatever the utility of the alternatives. In this specific case, notice that δ^{B} is an increasing transformation of δ^{A}. I show that in general the range and contrast pair is unique up to such transformations. Additionally, I propose normalized forms for the range contrast representation so that one can test, given two pairs $(F^{\mathrm{A}}, \delta^{\mathrm{A}})$ and $(F^{\mathrm{B}}, \delta^{\mathrm{B}})$, if and how their range and contrast differ.

Section 4 determines the behavioral implications of several restrictions on the contrast. The **contrast level** is the ratio $\alpha = {}^{\partial_2 \delta}/_{\partial_1 \delta}$ of the second to the first partial derivative of δ. In the examples of contrast given earlier, $\alpha^{\mathrm{A}} = \alpha^{\mathrm{B}} = -2$ is constant, whereas $\alpha^{\mathrm{C}} = {}^{1}/_{1+u_2^2}$ is not. The contrast level is a property of the model and does not depend on the range-contrast representation chosen, which explains why $\alpha^{\mathrm{A}} = \alpha^{\mathrm{B}}$. The contrast is also invariant to the scale of the cardinal utility. For instance, $\delta^{\mathrm{A}}(u_1, u_2) = u_1 - 2u_2$ and $\delta(v_1, v_2) = \exp(v_1) - 2\exp(v_2)$ have the same contrast level.

The case of **models with a constant contrast level** generates clear predictions. For instance, a negative contrast level - like α^{A} - implies that raising the utility of one alternative slows down the horse of the other and increases the probability of being chosen. A contrast level whose absolute value is smaller than 1 implies that decisions go faster as the utility of both alternatives increase. The argument also holds for models in which the contrast level vary but always satisfies the conditions - for instance, α^{C} that is always between -1 and 0. Models that have a range contrast representation with a linear contrast - like $(F^{\mathrm{A}}, \delta^{\mathrm{A}})$ - clearly have a constant contrast level. Conversely, models that have a constant contrast level also have a range contrast representation with a linear contrast. For instance, the model represented by $(F^{\mathrm{B}}, \delta^{\mathrm{B}})$ has a constant contrast level; even though δ^{B} is not linear. However, the model can also be represented by $(F^{\mathrm{A}}, \delta^{\mathrm{A}})$ and δ^{A} is linear.

The contrast of the Drift Diffusion Model imposes narrow restrictions. Representing a DDM by a race between two horses - each governed by its own scalar parameter d_i - is possible but a bit artificial since a single scalar $u_1 - u_2$ governs the whole distribution. I show that the models having this property are exactly those that admit a range contrast representation (F, δ) such that $\delta(u_1, u_2) = -\delta(u_2, u_1)$. Note that this is the case for the DDM since $\delta^{\mathrm{DDM}}(u_2, u_1) = u_2 - u_1 = -\delta^{\mathrm{DDM}}(u_2, u_1)$. I also show that the models with a constant contrast level that satisfy this property are exactly those with contrast level -1.

Section 5 is devoted to the econometric consequences of the previous results. The main focus here is on identification, under the assumption that the true data generating process satisfies the three axioms (Weak Scalar Parametrization), (Symmetry) and (Higher Utility, Faster Horse). I start with the **identification of the range**. The distribution of arrival time for each horse (T_1, T_2) is identified from the observed distribution of choice and response time (C, T). For instance, if $P(T \leq t \cap C = i) = e^{d_i}/(e^{d_1}+e^{d_2})\ (1 - \exp[-(e^{d_1} + e^{d_2})t]))$ for a given pair (d_1, d_2), then it must be that $P(T_i \leq t) = F^{\mathrm{A}}(t, d_i)) = 1 - \exp(-te^{d_i})$. If the distribution of choice and response time is observed under every possible condition, then the range F is identified up to an increasing transformation of the d_i.

The range matters for **revealing cardinal preferences**, but so does the contrast. Imagine that you observe the distribution of choice and response time under an unknown condition (u_1, u_2). I consider the extremely favorable case in which the model has a constant (but unknown) contrast level α and you happen to know the normalized range F such that the pair (F, δ) with $\delta(u_1, u_2) = u_1 + \alpha u_2$ represents the model. You analyze the data and note that $P(T_1 \leq t) = F(t, 30)$ and $P(T_2 \leq t) = F(t, 45)$. A contrast level $-1/2$ would imply that $(u_1, u_2) = (70, 80)$. A contrast level $-2/3$ would imply that $(u_1, u_2) = (108, 107)$. If you are interested in the relative monetary value of alternative 1, a slight change in the contrast level can lead to a sizable difference in willingness to pay.

The issue remains even when you observe the distribution of choice and response time for various (unobserved) utility of the alternatives and only try to reveal variations in utility. Starting from the previous example, imagine that you also observe the distribution of choice and response time in another unknown condition (u_1', u_2') for which $P(T_1 \leq t) = F(t, 40)$ and $P(T_2 \leq t) = F(t, 90)$. A contrast level 0 would imply that the utility of the second alternative increased much more than that of the first as compared to the initial setting - more precisely, $\Delta u_1 =:= u_1' - u_1 = 10$ and $\Delta u_2 := u_2' - u_2 = 45$. On the contrary, a contrast level -3 would imply that the utility of the second alternative increased much more than that of the first - more precisely, one can find $\Delta u_1 = {}^{35}/_{2}$ and $\Delta u_2 = {}^{15}/_{2}$ by solving the system of equations. This underscores the fact that the contrast level drives the interpretation of response time data in terms of utility and that a misspecified contrast level can lead to erroneous conclusions.

Fortunately, the **identification of the contrast level** is possible when there are small variations in utility are observed and two alternatives have a close value. For instance, consider a discrete choice experiment in which the same product is compared twice to the same outside option, but with a slightly lower price the second time. With the same notations as previously, this corresponds to the case $\Delta u_1 << 1$ and $\Delta u_2 = 0$. I also assume that the prices are chosen so that the decision-maker is almost indifferent between the two alternatives. Such a situation is observable as it corresponds to a choice probability of ${}^{1}/_{2}$ and means that $u_1 \simeq u_2$. I do not assume anymore that the range F is normalized, hence δ may not be linear. Still, for small variations Δu_1 in utility it is locally linear, hence the difference $\Delta d_1 := d_1' - d_1 = \delta(u_1 + \Delta u_1, u_2) - \delta(u_1, u_2)$ is close to $\Delta u_1 \times \partial_1 \delta(u_1, u_2)$ and $\Delta d_2 := d_2' - d_2 = \delta(u_2, u_1 + \Delta u_1) - \delta(u_2, u_1)$ is close to $\Delta u_1 \times \partial_2 \delta(u_2, u_1)$ - with $u_1 \simeq u_2$. Therefore, $\alpha \simeq \Delta d_2 \ / \ \Delta d_1$. As long as the distribution of choice and response times is observed, then the d_i and Δd_i are identified and so is α.

3 The range-contrast representation

In this section, I develop a theoretical framework to compare evidence accumulation models with each other. I start by clarifying my concept of model and recalling classical results on horse-race models. Then, I prove a representation result for a large class of models characterized by two axioms and containing the most prominent examples from the literature. Finally, I study the uniqueness of this representation under a third axiom and I derive normalized forms that can be used for comparing models.

The notion of model in Definition 1 is just a function that maps an input vector containing the utility of all the alternatives to an output joint distribution of choice and response times. This definition captures exactly what matters in a model for revealing cardinal utility from choice and response time. It is richer than the usual notion of stochastic choice function as it also accounts for response times. It abstracts from the numerous preference-irrelevant parameters that are part of the notion of models used in cognitive science. Each variation in these preference-irrelevant parameters correspond to a different model in my formalism.

My line of attack for comparing models builds on the argument that all models have a horse-race equivalent. This point has been first raised in Marley and Colonius [1992] and more recently in Jones and Dzhafarov [2014]. I revisit in Definition 2 the notion of horse-race model from Marley and Colonius [1992] to align it with my own notion of model. The usual equivalence result, Lemma 1, states in my setting that every model is equivalent to a horse-race model in which each alternative-specific horse depends on the cardinal utilities of the alternatives. For this reason, the assumptions and results of the paper are frequently formulated in terms of horses.

The rest of the section deals with the range-contrast representation of a model. A range-contrast representation (Definition 3) is a compact way to describe the horse-race equivalent of a model by its range, the set of possible horses for a given alternative, and its contrast, how the horse is determined by the cardinal utility of all the alternatives. I show in Theorem 1 that the models admitting a range-contrast representation are exactly those satisfying Axiom 1 (Symmetry) and Axiom 2 (Weak Scalar Parametrization). This class of models contains the DDM and LBA families of models as well as many of their variants.

The range-contrast representation is intended as a tool to compare models. Theorem 2 shows that, under Axiom 3 (Higher Utility, Faster Horse), the range-contrast representation is unique up to an increasing transformation of the contrast. Building on this result, Corollary 1 introduces a normalization procedure to align the contrasts of two models in order to compare the ranges and Corollary 2 another normalization procedure to align the ranges and compare the contrasts.

3.1 Preliminaries

Let me start by introducing some notations. A decision-maker faces $n \geq 2$ alternatives. Alternative i has a cardinal utility u_i to the decision-maker. The decision-maker makes a random choice $C \in [\![1, n]\!]$ after a random response time $T \in \mathbb{R}_+^*$. The joint distribution of (C, T) is entirely determined by the alternative-specific incidence functions $(H_i)_{1 \leq i \leq n}$ defined by

$$H_i(t) = P(C = i \ \cap \ T \leq t) \quad \text{for } t \in I \text{ and } 1 \leq i \leq n$$

Vectors are denoted by bold letters, such as $\mathbf{H} = (H_i)_{1 \leq i \leq n}$, $\mathbf{u} = (u_i)_{1 \leq i \leq n}$ the cardinal utility of each alternative. I note $X \sim F$ the fact that the random variable X has a cumulative distribution function F.

In the paper, the cardinal utility of the alternatives $\mathbf{u}$ affects the joint distribution of choice C and response time T. Instead of writing $C(\mathbf{u})$ and $T(\mathbf{u})$, I introduce $\mathbf{u}$ as a parameter in the alternative-specific incidence functions $H_i(t, \mathbf{u})$ which determines the law of (C, T) and I define everything in terms of $\mathbf{H}$.

Definition 1. *A **model** is a continuously differentiable function* $\mathbf{H} : \mathbb{R}_+^* \times \mathbb{R}^n \to (0, 1]^n$

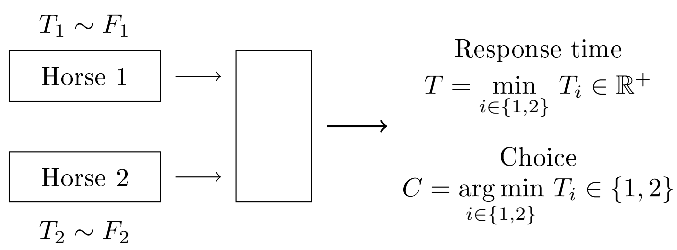


Figure 2: Illustration of a horse-race model for $n = 2$ alternatives

increasing in its first argument and such that, for all $u \in \mathbf{u} \in \mathbb{R}^n$*,*

$$\lim_{t \to 0} H_i(t, \mathbf{u}) = 0 \textit{ for all } 1 \leq i \leq n \textit{ and } \sum_{i=1}^{n} H_i(t, \mathbf{u}) \leq 1 \textit{ for all } t \in \mathbb{R}_+^*$$

The set of all models is denoted by $\mathcal{M}$*.*

This definition means that a model is a mapping from an input vector $\mathbf{u}$ of the cardinal utility of each alternatives to an output distribution of choice and response time (C, T) determined by its alternative-specific incidence functions $t \mapsto H_i(t, \mathbf{u})$ for $1 \leq i \leq n$. Note that the H_i are positive, hence each of the distribution $H_i(., \mathbf{u})$ for $1 \leq i \leq n$ has full support on $\mathbb{R}_+^*$ and the conditional variable $T \mid C = i$ admits continuous densities with respect to the Lebesgue measure for all $u \in \mathbb{R}^n$.

This definition focuses exclusively on the role of the cardinal utility of the alternatives. The other parameters - boundaries, threshold, starting points, non-decision time - that usually enter evidence accumulation models are understood as fixed in a model. Their influence over the predictions is not discussed here. Each value of this set of parameter gives corresponds to a specific model. Appendix A introduces formally two family of models, the DDM and the LBA model, which I use as examples throughout the paper.

Definition 2. *A* ***horse-race model*** *is a model* $\mathbf{H} \in \mathcal{M}$ *such that there exists a continuously differentiable function* $\mathbf{F} : \mathbb{R}_+^* \times \mathbb{R}^n \to (0, 1]^n$ *with*

$$\frac{\partial H_i}{\partial t}(t, \mathbf{u}) = \frac{\partial F_i}{\partial t}(t, \mathbf{u}) \prod_{j \neq i} (1 - F_j(t, \mathbf{u}))$$

The intuition behind this definition is that a horse-race model is a model such that, for all $\mathbf{u} \in \mathbb{R}^n$, there exist alternative-specific independent random variables $(T_i)_{1 \leq i \leq n}$ such that (C, T) satisfies

$$T = \min_{1 \leq i \leq n} T_i \quad \text{and} \quad C = \arg\min_{1 \leq i \leq n} T_i$$

Using these notations, F_i is just the probability distribution function of T_i, i.e. $F_i(t, \mathbf{u}) =$

$P(T_i \leq t)$ for $1 \leq i \leq n$. Using the vocabulary of competing risks in survival analysis, H_i is the incidence and F_i the hazard rate.

Modeling response and time data with horse-race models is attractive because of the natural extension to any number n of alternatives. In their seminal article, Marley and Colonius [1992] showed that the previously introduced notion of horse-race model imposes no constraint on the model. More precisely, they proved that any distribution $\mathbf{H}$ can be rationalized by a horse-race model under mild assumptions on $\mathbf{H}$ and they also precise under which condition the random response times for the same alternative are invariant across choice sets. Their proof was based on the following lemma, which can be traced back to Berman [1963].

Lemma 1 (Berman [1963]). *For any $(\mathbf{H}_i)_{1\leq ilen} : \mathbb{R}_+^* \times \mathbb{R}^n \to (0,1]$ that admits positive and continuous densities $(h_i)_i$ on $[0,+\infty)$, there exists T_1, ..., T_n independent random variable such that*

$$\begin{cases} H_i(t) & = P\left(\bigcap_{j\neq i}(T_i < T_j)\right) \text{ for all } i \in [\![1,n]\!] \\ C & = \arg\min_i T_i \end{cases}$$

Moreover, the cumulative distribution function $(F_i)_{1\leq i\leq n}$ of these alternative specific response times (T_i) are unique and characterized by

$$F_i(t) = 1 - \exp\left(\int_0^t \frac{h_i(s)}{1 - H(s)}\ ds\right)$$

This lemma warns us that the fact that a model is "horse-race" does not impose any constraint on the resulting joint distribution of response and time. However, this does not imply that this concept is useless. The horse-race approach provides useful concepts - such that the alternative-specific response times - based on which one can impose further constraints on the models. For instance, Townsend and Liu [2020] study the implications of having one alternative-specific response time first-order stochastically dominating another, or that the ratio of the probability distribution function of alternative-specific response time is increasing.

In the rest of this section, I use repeatedly the fact that every model can be written as a horse-race model.

3.2 Existence of a range-contrast representation

The notion of model introduced so far is so general that cross-model comparisons are numerically and analytically impossible. In terms of dimensions, a model $\mathbf{H} \in \mathcal{M}$ is a function from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$. For most of the models in the literature, we do not have a tractable analytical expression for this function. And even for binary choices ($n = 2$), in which case a model is a function $\mathbf{H} : \mathbb{R}^3 \to \mathbb{R}^2$, comparing two models by plotting their values for a range of inputs can hardly reflect all the dimensions in which they can differ.

I will introduce two general axioms that greatly simplify the task. The first one is (Symmetry).

Axiom 1 (Symmetry)**.** *A model $\mathbf{H} \in \mathcal{M}$ satisfies **(Symmetry)** if, for every $t \in \mathbb{R}^{+*}$, $u \in \mathbb{R}^n$ and $1 < i \leq n$,*

$$H_i(t, \mathbf{u}) = H_1(t, \tilde{\mathbf{u}}^{\mathbf{i}}) \text{ where } \tilde{u}^i = (u_i, u_2, \ldots, u_{i-1}, u_1, u_{i+1}, \ldots, u_n)$$

This first axiom is a very natural constraint when studying preferences. Since the ordering of the alternatives in the definition of the model is just a notational convenience, there is no reason to expect that one alternative is favored over another except because of its utility. In the case n = 2, the condition simply writes, for $t \in T$ and $(u_1, u_2) \in \mathbb{R}^2$,

$$H_2(t, (u_1, u_2)) = H_1(t, (u_2, u_1))$$

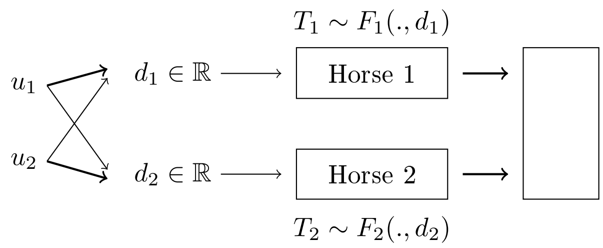
This axiom is not satisfied, for instance, by variants of the DDM model that feature either asymmetrical boundaries or a non-centered starting point. Such asymmetries are justified in the context of perceptual tasks, for which there is a natural "correct" answer, but not for value-based choice. Since my focus here is the use of evidence accumulation model to measure cardinal utility, it seems more appropriate not to introduce a priori a bias in favor of one alternative.

Under (Symmetry), a model $\mathbf{H} : \mathbb{R}^{+*} \times \mathbb{R}^n \to [0, 1]^n$ is determined by $H_1 : \mathbb{R}^{+*} \times \mathbb{R}^n \to [0, 1]$. This is much better than previously, but still difficult to plot or study analytically. The next axiom will further break down the dimensionality of the problem of comparing models.

Axiom 2 (Weak Scalar Parametrization)**.** *A model satisfies **(Weak Scalar Parametrization)** if there exists $d : \mathbf{u} \in \mathbb{R}^n \to d \in \mathbb{R}$ such that*

$$H_1(t, \mathbf{u}) = \widetilde{H}_1(t, d(\mathbf{u}))$$

This second axiom captures the fact that most evidence accumulation models in the literature can be described as governed by an intermediate object between the input alternative-specific utility and the output distribution of response and time, namely some notion of rate of evidence accumulation. A simple interpretation of this axiom is that the set of possible "horses" - i.e. alternative-specific response time distribution in the horse-race version of the



Notes: The cardinal utilities jointly determine a scalar drift rate d_i per alternative, which in turn determines the distribution of arrival time T_i of the corresponding horse. The empty rectangle on the right means that the distribution of choice and response times is determined from the distribution of arrival times of the horses following the horse-race decision rule depicted in Figure 2.

Figure 3: Illustration of the (Weak Scalar Parametrization) axiom for $n = 2$ alternatives

model - is a curve, in the sense that it can be smoothly parametrized by a scalar. One could think of the drift rate d_i as indicating the mood of horse i, which is itself determined by the utility u_i of the corresponding product as well as that of other products.

This axiom is satisfied in most models that feature a notion of drift rate. This is the case for the DDM model, in which the utility of the two alternatives matter only through the drift rate $u_1 - u_2$. This is also the case for the LBA model, in which the mean drift rate of each parallel process is affected by the utility u_1 of the corresponding alternative.

Example 1. *Both the DDM and LBA model satisfy (Weak Scalar Parametrization)*

I can now introduce the main object of this section, the range-contrast representation of a model.

Definition 3. *Let $\mathbf{H} \in \mathcal{M}$ be a model. A* ***range-contrast representation*** *of $\mathbf{H}$ is a pair (F, δ) where*

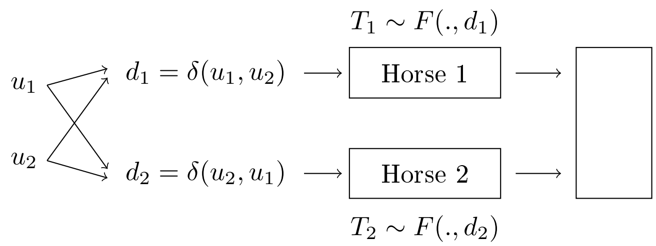
- $F : \mathbb{R}^{+*} \times \mathbb{R} \to [0, 1]$ *is continuously differentiable*
- $\delta : \mathbb{R}^n \to \mathbb{R}$ *is continuously differentiable*

such that, for all $t \in \mathbb{R}_+^*$, $\mathbf{u} \in \mathbb{R}^n$ and $1 \leq i \leq n$, $F_i(t, \boldsymbol{u}) = F(t, \delta(\bar{\mathbf{u}}^{\mathbf{i}}))$.

The intuition behind this definition is that F characterizes the range of possible choice and response time distributions with the model while δ describes how alternative-specific rates are combined to determine the outcome distribution.

$$F_1(t, \mathbf{u}) = F(t, \delta(u_1, u_2))$$
$$F_2(t, \mathbf{u}) = F(t, \delta(u_2, u_1))$$

Let me provide another interpretation in terms of horse race. The function F indicates how fast each horse will race depending on its mood d. The function δ indicates how the mood of horse i is determined by its own talent u_i and that of other horses $(u_j)_{j \neq i}$.



Notes: The contrast function δ determines a scalar drift rate d_i per alternative by processing symmetrically their cardinal utilities. The range function F determines the distribution of arrival time T_i of the horses from the corresponding drift rate d_i. The empty rectangle on the right means that the distribution of choice and response times is determined from the distribution of arrival times of the horses following the horse-race decision rule depicted in Figure 2.

Figure 4: Illustration of the range-contrast representation of a model for $n = 2$ alternatives

We have finally reached a representation of a model that is concise enough to allow for cross-model comparisons. As it happens, $F : \mathbb{R}_+^* \times \mathbb{R}^2 \to [0,1]$ can easily be visualized with a heat map. Note that the dimensionality of the range does not depends on n. What depends on n is the dimension of the contrast $\delta : \mathbf{R}^n \to \mathbf{R}$. However, only two contrasts δ have been considered in the literature so far: $\delta(u_1, u_2) = u_1$ and $\delta(u_1, u_2) = u_1 - u_2$.

Example 2.
- *The LBA model admits a range-contrast representation* (F, δ) *with* $\delta(u_1, u_2) = u_1$.
- *The DDM model admits a range-contrast representation* (F, δ) *with* $\delta(u_1, u_2) = u_1 - u_2$.

To extend the previous equine analogy, in the LBA model, the mood of the horse depends only on its own talent, whereas In the DDM model, the mood of the horse depends on its on talent but is negatively affected by the talent of others. Because the function δ determines the extent to which the mood of one horse can be affected by that of the others, I call it "contrast".

The following theorem is the first novel result of this paper. It shows that the two previous axioms characterize the class of model that admit a range-contrast representation.

Theorem 1. *A model satisfies (Symmetry) and (Scalar Parametrization) if and only if it admits a range-contrast representation.*

The proof consists mostly in considering the equivalent horse-race version of the model, simplifying its expression using (Symmetry) and (Scalar Parametrization). A detailed proof is available in appendix.

The two previously introduced axioms define a wide class of models, robust that encompasses most of the existing examples from the literature. Theorem 1 shows that these models all admit a range-contrast representation, and we have argued that such a representation is very convenient for performing cross-model comparisons.

Before performing actual cross-model comparisons, a last issue needs to be solved. While each range-contrast representation determines a unique model, the converse does not hold: a model always has several range-contrast representations. For instance, notice that if (F, δ) is a range-contrast representation of a model, so is $(F \circ \phi^{-1}, \phi \circ \delta)$ for any bijection $\phi : \mathbb{R} \to \mathbb{R}$. Therefore, two contrast-range representations can have a different range and contrast even though they represent the same model. The following subsection explains how to overcome this challenge.

3.3 Uniqueness of a normalized range-contrast representation

So far, we have only proved the existence of range-contrast representations for models. As we have previously noted, such representations are never unique, since the following transformation does not change the model for every continuous bijection $\phi : \mathbb{R} \to \mathbb{R}$.

$$(F, \delta) \longrightarrow (F \circ \phi^{-1}, \phi \circ \delta)$$

In order to perform cross-model comparisons, it is thus essential to find a normalized form for the range-contrast representation of a model that will be unique. To do so, I introduce the (Higher utility, Faster Horse) axiom and show that models that satisfy this axiom in addition to the two previous ones (Symmetry) and (Weak Scalar Parametrization) have a unique range-contrast representation up to a simple normalization.

Axiom 3 (Higher Utility, Faster Horse)**.** *A model satisfies (Higher Utility, Faster Horse) if, for all $t \in \mathbb{R}$, $\mathbf{u} \mapsto F_1(t, \mathbf{u})$ is increasing in u_1 and non-increasing in u_2.*

Thus axiom means that the alternative-specific response time T_i (the arrival time of horse i) decreases with u_1 and is non-increasing in u_2 in the sense of first-order stochastic dominance. In other words, as the utility of an alternative increases, the corresponding horse becomes faster. This has two immediate implications:

Proposition 1. *If a model satisfies (Symmetry), (Weak Scalar Parametrization) and (Higher utility, Faster Horse), then*

- *the choice probability $P(C = i)$ for alternative i is increasing in u_i.*
- *the model admits a range-contrast representation (F, δ) such that $F(t, d)$ is increasing in d for all $t \in \mathbb{R}^{+*}$ and $\delta(u_1, u_2)$ is increasing in u_1 and non-increasing in u_2.*

Once again, this axiom is quite intuitive and satisfied by a large class of existing models, such as the ones that we have considered so far. Proving that the axiom is satisfied is not obvious for models that are usually not presented as race models. When it is the case, this property can also be easily tested by plotting the range of a range-contrast representation of the model.

Example 3. *Both the DDM and LBA model satisfy (Higher Utility, Faster Horse).*

Now that this new axioms has been introduced, it is possible to prove uniqueness results. The key step is the following normalization theorem.

Theorem 2 (Uniqueness up to an increasing transformation)**.**

- *If a model satisfies (Symmetry), (Weak Scalar Parametrization) and (Higher utility, Faster Horse), then it admits a range-contrast representation* (F, δ) *with* δ *increasing in its first argument.*

- *If a model admits a range-contrast representation* (F, δ) *with* δ *increasing in its first argument, then* $(\bar{F}, \bar{\delta})$ *is a range-contrast representation of the model if and only if there exists* ϕ *increasing such that*

$$\begin{cases} \bar{\delta} = \phi \circ \delta \\ \bar{F} = F \circ \phi^{-1} \end{cases}$$

This important result will allow us to design normalization procedures to perform cross-model comparisons. Note that a single normalization method may not be at the same time indicative of the difference in terms of range and in terms of contrast between models. In what follows, I propose two normalizations for the range-contrast representation, one for comparing the range and one for comparing the contrast across models.

Corollary 1 (Normalization of the range at a specific time)**.** *If a model satisfies (Symmetry), (Weak Scalar Parametrization) and (Higher Utility, Faster Horse), then for every* $t_0 \in \mathbb{R}^{+*}$*, it admits a unique range-contrast representation* (F, δ) *such that*

$$F(t_0, \delta(u_1, u_2)) = \delta(u_1, u_2) \text{ for all } (u_1, u_2) \in \mathbb{R}^2$$

This normalization procedure is specially useful to compare the range of two models. The idea is to "align" their contrast at a specific point $t_0 \in \mathbb{R}^{+*}$ in time and observe how the predictions of the two models diverge as we consider larger or smaller response times.

Corollary 2 (Normalization of the contrast)**.** *If a model satisfies (Symmetry), (Weak Scalar Parametrization) and (Higher Utility, Faster Horse), then it admits a unique range-contrast representation* (F, δ) *such that* $\delta(u_1, 0, \ldots, 0) = u_1$ *for all* $u_1 \in \mathbb{R}$.

This normalization procedure is specially useful to check whether the model admits a range-contrast representation (F, δ) in which δ is linear in the utility vector (u_1, u_2). If such a δ does exist, then it corresponds to the contrast of the normalized representation in the sense of Corollary 2. The following subsection is devoted to the study of models having this property.

4 The contrast level

This section examines the contrast level, a key elasticity that plays a pivotal role when revealing cardinal preferences using models of choice and response time. I first define the contrast level in a general context, discussing the robustness of the notion and its behavioral implications. Then, I make these results more precise on the class of models having a constant contrast level, to which virtually all the evidence accumulation models in the literature belong. Finally, I offer a characterization of the models whose contrast function resembles that of the DDM.

The contrast level is a real number measuring the extent to which the horse corresponding to one alternative is affected by the utility of the other alternatives. In all generality, the contrast level is a function, introduced in Definition 4, of the utility of each alternative. The contrast level depends only on the contrast function, but it is an intrinsic property of the model, in the sense that it does not depends on which range-contrast representation is chosen. Proposition 2 shows that a model makes specific predictions when its contrast level is restricted to a certain range of values. I study in Proposition 3 the invariance of the contrast level when the scale of the cardinal utility varies.

The contrast level is constant in virtually all the models considered in the literature that satisfy the axioms of Section 3. All original horse-race models have a contrast level of 0, all the models of the DDM family have a contrast level of -1. The importance of this elasticity, whose main implications are listed in Proposition 2, has not been explored in earlier research. Proposition 4 shows that the models for which the contrast level is constant are exactly those that admit a range-contrast representation with a linear contrast function. From a theoretical perspective, there is no reason why the literature has considered only two possible values for the contrast level.

Making use of the previous analytical framework, I am able to characterize some specificities of the DDM within the wider class of models introduced so far. The DDM family of models satisfy Axiom 4 (Strong Scalar Parametrization), which means that they are parametrized by a single scalar drift, Theorem 3 shows that the models that satisfy (Symmetry), (Strong Scalar Parametrization) and (Higher Utility, Faster Horse) are exactly those whose contrast function is anti-symmetric. This non-trivial result has two direct implications. The first one, Corollary 3, is that there are no such models when there are more than three alternatives. The second, Corollary 3, is that the models that have a constant contrast of -1 are exactly those that have a linear contrast function and satisfy (Strong Scalar Parametrization).

4.1 A robust concept with clear behavioral implications

I consider a model M that satisfies the three axioms (Symmetry), (Weak Scalar Parametrization) and (Higher Utility Faster Horse) introduced in the previous section. We know from Theorems 1 and 2 that such a model admits a range-contrast representation that is unique up to an increasing transformation.

Definition 4. *The contrast level of a model M with a range-contrast representation (δ, F) is the function α defined as*

$$\begin{array}{rccc} \alpha : & \mathbb{R}^{\ltimes} & \longrightarrow & \mathbb{R} \\ & \mathbf{u} & \longmapsto & \frac{\frac{\partial \delta}{\partial u_2}(\mathbf{u})}{\frac{\partial \delta}{\partial u_1}(\mathbf{u})} \end{array}$$

It is well-defined and does not depend on the range-contrast representation (δ, F) chosen.

Imposing constraints on the contrast level has some immediate behavioral implications. The simplest way to illustrate this is the case of two products A and B of equal value, that is to say $n = 2$ and $\mathbf{u} = (u, u)$ with $u \in \mathbb{R}$.

Proposition 2 (Behavioral implications)**.** *For two alternatives A and B of equal value :*

- *If the function α is always negative (resp. positive), then increasing the utility of B makes the horse associated to A slower (resp. faster).*
- *If the function α is always strictly smaller than -1, then increasing the utility of both A and B by the same amount makes choices slower.*

The contrast level function α is not only robust to the range-contrast representation chosen, but also to the scale of the cardinal utility, at least in the case of two products of equal value.

Proposition 3. *[Invariance to the scale of the cardinal utility]*

Given $u \in \mathbb{R}^n$, introduce a smooth increasing transformation $\phi : \mathbb{R} \to \mathbb{R}$ and define $\mathbf{v} = (\phi^{-1}(u_i))_{1 \leq i \leq n}$. Let (δ, F) be a range-contrast representation of a model M_U and (δ_ϕ, F) a range-contrast representation of the model M_V that is perfectly equivalent to M_U after rescaling - hence $\delta_\phi(a_1, \ldots, a_n) := \delta(\phi(a_1), \ldots, \phi(a_1))$. Denoting α_U the contrast level *of U and α_V that of M_V, we have*

$$\alpha_V(\mathbf{v}) = \alpha_U(\mathbf{v}) \times \frac{\phi'(u_2)}{\phi'(u_1)} \textit{ for all } \mathbf{u} \in \mathbb{R}^n$$

In particular, wherever two of the alternatives are of equal utility, the contrast level does not depend on the scale.

Proposition 3 implies that the results of Proposition 2 are independent of the scale of the cardinal utility. In the following subsection, I focus on models for which the contrast level is constant. In this case, the relation obtained in Proposition 3 evaluated in $0_{\mathbb{R}^n}$ ensures that the constant contrast level remains the same after re-scaling the cardinal utility.

4.2 Models with a constant contrast level

The models with a constant contrast level are exactly those that admit a range-contrast representation (δ, F) in which δ is linear. Let me formalize this statement.

Definition 5. *A contrast function $\delta; \mathbb{R} \to \mathbb{R}$ is linear-in-utility if there exists $a, b \in \mathbb{R}$ such that $\delta(u_1, u_2) = a\ u_1 + b\ u_2$*

This result offers a characterization of the models that have a constant contrast level, and also a characterization of the contrast level itself.

Proposition 4. *A model admits a range-contrast representation with a linear-in-utility contrast if and only if it has a constant contrast level.*

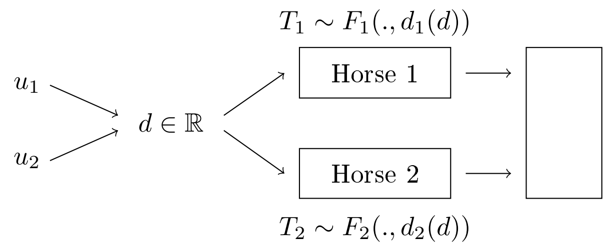
If it is the case, then there exists a unique $\alpha \in \mathbb{R}$ such that the model admits a range-contrast representation (F, δ) with $\delta(u_1, u_2) = u_1 + \alpha u_2$. This α is the (constant) contrast level of the model.

Since most of the models in the literature admit a range-contrast representation with a linear-in-utility contrast, it is possible to compare their contrast level. Actually, only two contrast levels are featured in existing models.

Example 4. *In the following cases, the contrast level is constant*

- *it is equal to -1 for the DDM family of models,*
- *it is equal to 0 for originally horse-race models such as the LBA*

The fact that such a key behavioral parameter is set to an arbitrary value by classical models is problematic. The most appropriate level of contrast is an empirical question that has not been addressed yet in the literature and has important implications when it comes to revealing cardinal preferences from choice and response time data.



Notes: The cardinal utility of each alternative jointly determine a single scalar drift rate d, which in turn determines the distribution of arrival time T_i for all the horses. The empty rectangle on the right means that the distribution of choice and response times is determined from the distribution of arrival times of the horses following the horse-race decision rule depicted in Figure 2.

Figure 5: Illustration of the (Strong Scalar Parametrization) axiom for $N = 2$

One may argue that a contrast level of -1 is theoretically attractive because (1) it seems natural to have a negative contrast and (2) the value of -1 creates some sort of anti-symmetry between the alternatives. In particular, a contrast level of -1 implies that $\delta(u_2, u_1) = -\delta(u_1, u_2)$ for the linear-in-utility range-contrast representation (F, δ), hence the characteristics of one horse are entirely determined by the characteristics of the other. The following subsection will clarify the implications of such an assumption.

4.3 Characterizing the contrast of the DDM

When the (Weak Scalar Parametrization) axiom was introduced in Section 3, I pointed out that the DDM satisfies a much stronger property. In the DDM model, the output distribution depends on to the input utility vector through a single scalar, whereas (Weak Scalar Parametrization) allows one scalar per horse. The following axiom captures this specificity of DDM-like models.

Axiom 4. *[Strong Scalar Parametrization]*

*A model satisfies (**Strong Scalar Parametrization**) if there exists $(\bar{H}_i)_{1 \le i \le n} : \mathbb{R} \times \mathbb{R}^n \to [0,1]$ and $d : \mathbb{R}^n \to \mathbb{R}$ such that, for all $1 \le i \le n$, $t \in \mathbb{R}$ and $\mathbf{u} \in \mathbb{R}^n$,*

$$H_i\left(t, \mathbf{u}\right) = \bar{H}_i\left(t, d(\mathbf{u})\right)$$

The following proposition clarifies the relation between the axioms introduced so far. The proof is straightforward given the definition of the axioms and the fact that the LBA model does not satisfy (Strong Scalar Parametrization).

Proposition 5. • *(Strong Scalar Parametrization) implies (Weak Scalar Parametrization)*

- *(Weak Scalar Parametrization), (Symmetry) and (Higher utility, Faster Horse) do not* *jointly imply (Strong Scalar Parametrization)*

In a closely related paper, Alós-Ferrer et al. [2021] introduce a wide class of models of choice and response times for binary choice using chronometric functions. All these models also satisfy the (Strong Scalar Parametrization) axiom.

The remainder of this section investigates what the (Strong Scalar Parametrization) axiom imposes on the range-contrast representation of a model. Theorem 3 characterizes the models that satisfy the same axioms as Theorem 1 except that (Weak Scalar Parametrization) has been replaced by (Strong Scalar Parametrization).

Theorem 3. *A model satisfies (Strong Scalar Parametrization), (Higher Utility, Faster Horse) and (Symmetry) if and only if it admits a range-contrast representation (F, δ) such that, for all $u_1, u_2 \in \mathbb{R}$, $\delta(u_2, u_1) = -\delta(u_1, u_2)$*

The class of model that satisfy these three axioms is the class of models which have a range-contrast representation in which the contrast function is anti-symmetric. The proof makes use of a characterization of the continuous involutions of $\mathbb{R}$ - a mathematical result commonly attributed to Babbage [1815]. My theorem has two important implications.

First, it makes it possible to characterize the family of models that have the same contrast as the DDM model. Corollary 2 states that they are exactly the models that satisfy the assumptions of Theorem 3 and admit range-contrast representation with a linear-in-utility contrast.

Corollary 3. *A model satisfies (Strong Scalar Parametrization), (Higher Utility, Faster Horse), (Symmetry) and admits a range-contrast representation (F, δ) in which δ is linear-in-utility if and only if it admits a range-contrast representation $(\bar{F}, \bar{\delta})$ such that*

$$\delta(u_1, u_2) = u_1 - u_2$$

Second, it suggests that (Strong Scalar Parametrization) is not a suitable axiom for modeling non-binary choices. The intuition is that for $N \geq 3$, the anti-symmetry of the contrast function is incompatible with the symmetrical role played by the utility of the other alternatives.

Corollary 4. *There is no non-trivial model that satisfies (Strong Scalar Parametrization), (Higher Utility, Faster Horse) and (Symmetry) for $N \geq 3$*

5 Econometrics implications

This section presents the implications of the previous theoretical results for the econometrics of choice and response times. I start by general identification conditions applicable to the large class of models presented in Section 3. Then, I show how to visualize the difference in range between the DDM and LBA models. Finally, I show how to calibrate the contrast level of a model while keeping the range constant;.

I address two important questions about identification. First, what do we need to know about the model in order to retrieve the cardinal utility of the alternatives from choice and response time? Second, what can we infer about the model from observable data? Example 5 gives partial negative results to both questions. It shows that it is impossible to retrieve the cardinal utility of the alternatives without knowing the contrast of the model - even under the assumption that the contrast level is constant. It also shows that it is impossible to determine the contrast of the model - even just the contrast level - using choice and response time data only.

Identification becomes interesting if we observe more than just choice and response time. I derive results about model identification (Proposition 6) under three conditions : when only choice and response time are observed (Assumption 1), when the utility of the alternatives are observed (Assumption 2) and when some variations in the utility of the alternatives are observed (Assumption 3). The main result is Proposition 6, which implies that the range is identified from choice and response time only, whereas the contrast level is identified from observable variations in utility.

Then, I illustrate the strength of my theoretical results by comparing the range of the LBA and DDM models. To motivate this exercise, let me suggest an analogy with random utility models. We know that the only difference between the probit and logit amounts to the distribution of their noise terms - normal or Gumbel - and these distributions can easily be represented on the same graph in order to understand where the gaps emerge. In a similar vein, Figure 6 illustrates the range of the LBA and DDM models and how they deviate from each other across time and drift rate. The construction of such a graph relates to the normalized range contrast representation for range comparison that was introduced in Corollary 1.

Finally, I show how to calibrate the contrast level of an existing race model. I introduce my methodology and propose variants depending on what is observable about the utility of the alternatives in applications. I then evaluate the performance of the variants in a Monte Carlo calibration exercise, in which I choose an LBA as my baseline race model and calibrate it on data generated by a DDM model.

5.1 Identification

The econometric analysis of choice and response time typically starts by selecting a specific class of models to analyze the data. Formally, the econometrician selects a family $\{M_\theta \mid \theta \in \mathcal{P}\} \subset \mathcal{M}$ of models parametrized by some parameter $\theta \in \Theta \subseteq \mathbb{R}^p$. The variable θ stands for the familiar parameters other than the drift rate associated to the family of evidence accumulation models considered, such as the threshold, the distribution of the starting point... As I stressed in the beginning of Section 3, with my definition of model as a mapping from an input vector of alternative-specific utility to an output distribution of choice and response time, each value of the parameters θ may correspond to a different model M_θ. Here, my objective is not to focus on the identification of the parameters θ for a given family of models such as the DDM or LBA, but rather on identification of the range and contrast in general. Therefore, I will assume that θ is identified, in the sense that the following function is injective

$$\begin{array}{rcl} M:\Theta & \longrightarrow & \mathcal{M} \\ \theta & \longmapsto & M_\theta \end{array}$$

I will also assume that the models in $M(\Theta)$ satisfy the three axioms from Section 3, namely (Symmetry), (Weak Scalar Parametrization) and (Higher Utility Faster Horse). The models need not have a constant contrast level, and I will explicitly mention which of my identification results rely on this additional assumption.

The data available to the econometrician may depend on the circumstances. I will always assume that the data contains enough information to identify the output choice and response time distribution, possibly under different conditions. Regarding the information available about the cardinal utility of the alternatives, I will consider three cases:

A first case where the data contains no information about how the cardinal utility of the alternatives changes across conditions. This is often the case in the existing literature in cognitive science.

Assumption 1 (The cardinal utility is totally unobserved)**.**

$$\phi = \{\mathbf{H}(.,\mathbf{u}) \mid \mathbf{u} \in \mathbb{R}^n\}$$

A second case where the varying cardinal utility of the alternatives across conditions is fully known. This ideal setting can be obtained in experiments with induced preferences. Even though it may not be useful in itself - what is the point of revealing cardinal preferences from choice and response time if you know them already - it should be understood as a first best for model calibration.

Assumption 2 (The continuously varying cardinal utility is observed)**.**

$$\phi = \{(\mathbf{u}, \mathbf{H}(., \mathbf{u})) \mid \mathbf{u} \in \mathbb{R}^n\}$$

A third case where the cardinal utility of the alternatives in itself is not known, but we know some of the continuous variations of the cardinal utility across condition. This setting is particularly relevant in all applications involving an observable and continuous characteristic like prices.

Assumption 3 (Some continuous variations of the cardinal utility are observed)**.**

$$\phi = \{(\Delta\mathbf{u}, \mathbf{H}(., \mathbf{u} + \Delta\mathbf{u})) \mid \Delta\mathbf{u} \in V \subseteq \mathbb{R}^n\} \textit{ for some neighborhood } V \textit{ of } 0_{\mathbb{R}^n}$$

Now that the relevant set of assumptions have been defined, here comes the main result regarding identification.

Proposition 6 (Identification)**.** *The range of the model is identified under Assumption 1. The contrast of the model is identified under Assumption 2, but not under Assumptions 1 or 3. The contrast level of the model is identified under Assumption 2 or 3, but not under Assumptions 1.*

I conclude this subsection by an explicit counter-example for the non-identification results.

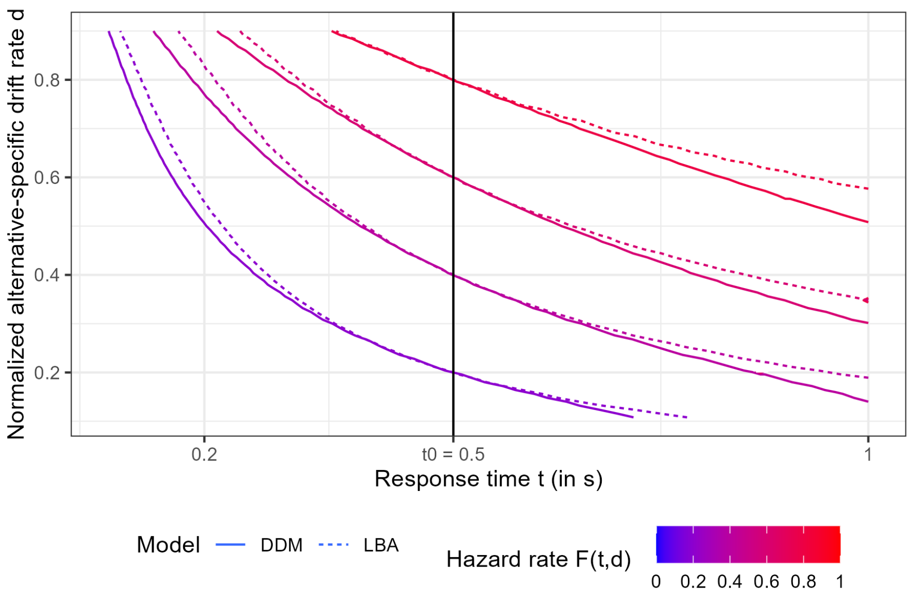
Example 5. *Consider a model whose range is $F(t, d) = 1 - \exp(-dt)$. Imagine that the observed choice and response time distribution is such that the alternative-specific horses are distributed according to $F_1(.) = F(., 30)$ and $F_2(.) = F(., 45)$. Here are two pairs of contrast levels and values of the cardinal utility of each alternatives that rationalize the data :*

- $\alpha = -1/2$ *and* $(u_1, u_2) = (70, 80)$.
- $\alpha = -2/3$ *and* $(u_1, u_2) = (108, 117)$.

5.2 Comparing the range of two models

In this subsection, I show how to visually compare the range of two models. The approach derives directly from Corollary 1. I apply it to the DDM and LBA models and discuss the implications of the results.

Figure 6 depicts the values of the normalized range F for a DDM and a LBA calibrated on the same dataset. I normalize the values of F using the uniqueness given by the theorem with $t_0 = 0.5$. Because of this normalization, we have $d \in [0, 1]$ and $F_{\text{LBA}}(0.5, d) = F_{\text{LBA}}(0.5, d) =$



Notes: The reference time $t_0 = 0.5$ is symbolized by a vertical line. Since the (normalized) range F is a function $\mathbb{R}^2 \to [0,1]$, I represent it by its level curves, i.e the set of argument for which it takes a given value. Here these levels are 0.2, 0.4, 0.6 and 0.8 and each has an associated curve. By construction, the normalized range of the LBA and DDM are the same at the reference time. These normalized range also satisfy $F(t_0, d) = d$ for $d \in [0,1]$, which imposes where the level curves intersect the vertical line. Differences in the range of the two models are reflected in the divergence of their level curves.

Figure 6: Normalized values of F for the DDM and LBA model

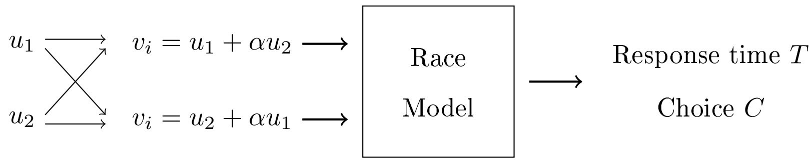
d for every $din[0,1]$. I plot the set of points for which the normalized value of F equals 0.25, 0.5 or 0.75. Overall, the dash and full curves are very close from one another, in particular for the small response times during which most of the decisions are made. Numerical instability explains the anomalies in the bottom right corner. More details on the numerical simulations are available in the appendix.

5.3 Calibrating the contrast level of an existing model

In this subsection, I show how to calibrate the contrast level of an existing race model in a binary choice task. I first present the method, then calibrate a LBA model on data generated by a DDM model, and finally discuss some practical applications of this method.

The motivation for re-using an existing model is simple. Finding a model with an appropriate range requires an extensive amount of choice and response time data. Calibrating the contrast level requires much less data but some variations in utility must be observed. This suggests a natural division of labor in which cognitive scientists would design and calibrate models that have a correct range, while economists would adjust the contrast level of these models to their setting. I perform both steps here, but the novelty is the second part.

Starting from a race model is a natural choice. First, because race models have a contrast



Notes: The race model represented by the square normally receives as input the utility u_i of each alternative. Instead, we feed the model with the v_i that are obtained from the u_i and some external parameter α. The resulting model has the same range as the initial race model, but a constant contrast level of α.

Figure 7: Introducing a contrast level parameter in an existing race model

level of 0, so the parameter α is exactly the contrast level of the calibrated model. Second, because the alternative would be to use a DDM model and my approach is not directly applicable in this case. Third, because the log-likelihood of the most popular race models in the literature has a tractable analytical expression, which facilitates the calibration.

Theoretically, the calibration could consider all the possible models with a constant contrast level that have the range of the initial model. Formally, the existing model gives a range F. We want to find all the contrast δ such that (F, δ) has a constant contrast level. Proposition 4 shows that these are exactly the δ such that $\delta(\mathbf{u}) = \phi(u_1 + \alpha \sum_{i \neq 1} u_i)$ where $\alpha \in \mathbb{R}$ and $\phi : \mathbb{R} \to \mathbb{R}$ is increasing. For instance, assuming that you find a race model with range F that can rationalize all the distributions of choice and response time of a DDM, then there exists an increasing function ϕ such that the model (δ, F) with $\delta(u_1, u_2) = \phi(u_1 - u_2)$ is exactly the DDM model. However, it may be difficult in practice to explore all the possible functions ϕ. Insofar as the variations in utility are not too large, one can rely on the linear approximation $\delta(u_1, u_2) = d + s(u_1 + \alpha u_2)$.

I propose to calibrate the contrast level under this local approximation and the additional assumption that the two alternatives have a similar value. Note that this last point is easily testable as it implies that the choices are equally split between the two alternatives. If the utility of u_1 and u_2 are close from each other, the same local approximation of δ can be used in (u_1, u_2) and (u_2, u_1). I introduce the free parameters d for the offset, s for the slope and α for the contrast level. Instead of feeding the race model directly with the utility u_i of each alternative, I first compute $v_i = d + s(u_i + \alpha u_j)$ for $i \neq j$ and then feed the model with the v_i, as depicted on Figure 7. This determines a distribution of the data conditional on the free parameters d, s and α as well as the (partially) observed utility u_i of the alternatives. The calibration consists in finding the most appropriate value of the contrast level parameter α given the data.

According to Proposition 6, some variations in utility must be observed for the contrast level to be identified. I consider two cases. The first case is when the u_i are directly observed, for instance in lab experiment with induced values or in a Monte Carlo calibration exercise.

Configuration	Parameters	Sample size	Estimated α
The $(u_i)_{i\in\{1,2\}}$ are observed	A, b, σ_v	10,000	0.96

Table 1: Results of the Monte Carlo contrast calibration exercise

Then, the u_i are not a parameter to be calibrated but part of the data. In this case, the structural equation $v_i = d + s(u_i + \alpha u_j)$ makes perfect sense as it is written. The second case is when small variations in the u_i are observed, for instance in a choice experiment in which the price of the two goods offered can be manipulated. Then $u = u^{\mathrm{REF}} + \Delta u$ where only Δu is observed and varies and u_i^{REF} is a constant. In such as case, replacing u_i by $u^{\mathrm{REF}} + \Delta u$ in the equation $v_i = d + s(u_i + \alpha u_j)$ does not make much sense, since both d and u^{REF} are unobserved. Since the unobserved reference utility u^{REF} can be captured by the offset d_i, one can simply use the structural equation $v_i = d + s(\Delta u_i + \alpha \Delta u_j)$. If the price of one of the two goods does not change, say good 1, then the equations simplify to $v_1 = d + s\alpha\Delta u_j$ and $v_2 = d + s\Delta u_j$.

I illustrate this approach in a Monte Carlo calibration exercise. When generating the data, I vary the utility of the alternative in the following way : I use LBA model whose range is consistent with this data using the same approach as in Subsection 5.2 - see Appendix C for more details.

- **Data generation.** I generate data using a DDM model. The preference-irrelevant model parameters like the threshold and the standard deviation of the Brownian motion are set to arbitrary values. I vary the utility (u_1, u_2) of the alternatives: u_1 is drawn from a standard normal distribution and u_2 is sampled uniformly from $\{-1, 0.2, 0.8\}$. The drift rate d of the DDM is $u_1 - u_2$.

- **Range calibration.** I follow the methodology from Appendix C to obtain a LBA model that is able to reproduce the distribution of choice and response time.

- **Contrast level calibration when variations in utility are observed.** I calibrate the contrast level of the model, assuming that u_1 and u_2 are observed. The structural equations write

$$\begin{cases} v_1 = d + s(u_1 - \alpha u_2) \\ v_2 = d + s(u_2 - \alpha u_1) \end{cases}$$

Table 1 presents the results of the calibration.

6 Discussion and conclusion

This paper introduces a novel theoretical framework to study models of choice and response times. Focusing on the role of the cardinal utility of the alternatives, I bring a fresh perspective to old arguments - such as the equivalence with horse-race models - and long-lasting debates on differences between models.

The theoretical results of the paper apply to a wide class of models. This class is characterized by three simple axioms shared by the LBA and DDM models as well as many of their variants. I introduce a formalism, based on the notion of range-contrast representation, which makes these models analytically comparable with each other. The contrast determines alternative-specific drift rates from the utility of all alternatives. The range determines the set of distributions of choice and response times that can be generated from the data. The range-contrast pair characterizes the model.

The contrast level is the main conceptual contribution of this paper. It is a key elasticity that appears in these models and drives their behavioral predictions. The higher the contrast level, the more the response times associated to an alternative are sensitive to the cardinal utility of the others. All the models considered in the literature have a constant contrast level - namely, -1 in the DDM and 0 in the LBA and other "horse-race" models. This choice seems arbitrary, for three reasons. First, plenty of models satisfying the three previous axioms do not have a constant contrast level. Second, the qualitative predictions of the model are reasonable when the contrast level is a constant between -1 and 0. Third, a constant contrast level of -1 is impossible in models of non-binary choice.

The econometric implications of these theoretical results are fundamental. The existing literature, which calibrated its models without any information about the cardinal utility of the alternatives, could only identify the range, but not the contrast. This explains why the contrast level, in spite of its central role for the measure of cardinal utility, has been mostly overlooked. I also show that the contrast level can be introduced as a free parameter and identified from choice and response time data in presence of observed variations in utility. This last assumption is easily satisfied in potential economic applications, such as choice experiments or choice sets with known variations in prices for a given item. I also compare the range-contrast representation of the LBA and DDM model and find that their range are quite comparable. This suggests that a LBA model modified to have a contrast level of -1 is roughly equivalent to a DDM model. Therefore, the LBA model with a flexible contrast level can be seen as admitting both the usual LBA and the DDM as particular cases and is not limited to binary choices. These results are of immediate methodological relevance to the burgeoning applications of evidence accumulation models in economics.

These theoretical advances pave the way for — and indeed demand — more empirical investigation. The most pressing issue is to know more about the value of the contrast level, whether it varies across individuals and experimental settings and how it relates to other behavioral patterns already measured in the literature. Then, the large literature comparing existing models of choice and response time based on goodness of fit may find useful to plot and compare the range of models calibrated on the same actual data - just as this paper has done for the LBA and DDM - to better understand the relation between their drift rates and their predictions. In parallel, it would be interesting to apply models of choice and response times to existing economic datasets involving non-binary choices and evaluate their effect on inference.

There are still several theoretical hurdles to clear before models of choice and response times can be used in most economic applications. One key issue is that of varying size of the choice set. This is a challenging point because we know extremely little about the effect of varying the number of alternatives, except for Hicks law, which states that the average response time increases logarithmic ally with the number of alternatives. It could be interesting to derive the implications of Hicks law for the contrast level depending on the number of alternatives, and test it empirically. Another approach would be to provide a micro foundation for intermediary values of the contrast level by making specific assumptions regarding how the attention of the decision-maker is attributed to alternatives and derive the implications of such a decision process to varying choice sets.

Overall, this paper aims to bridge the gap between the widespread use of choice and response time models by cognitive scientists and the cautious attitude of economists that prefer to thoroughly understand the implications of their models before using them in normative applications.

References

Khai Xiang Chiong, Matthew Shum, Ryan Webb, and Richard Chen. Combining Choice and Response Time Data: A Drift-Diffusion Model of Mobile Advertisements. *Management Science*, May 2023. ISSN 0025-1909. doi: 10.1287/mnsc.2023.4738.

Renato Berlinghieri, Ian Krajbich, Fabio Maccheroni, Massimo Marinacci, and Marco Pirazzini. Measuring utility with diffusion models. *Science Advances*, 9(34):eadf1665, August 2023. doi: 10.1126/sciadv.adf1665.

Carlos Alós-Ferrer, Ernst Fehr, and Nick Netzer. Time Will Tell: Recovering Preferences When Choices Are Noisy. *Journal of Political Economy*, 129(6):1828–1877, June 2021. ISSN 0022-3808. doi: 10.1086/713732.

Carlos Alos-Ferrer and Michele Garagnani. Improving Risky-Choice Predictions Using Response Times. *Journal of Political Economy Microeconomics*, page 728666, November 2023. ISSN 2832-9368, 2832-9376. doi: 10.1086/728666.

Xinwei Li and Prateek Bansal. The Importance of Response Time in Preference Elicitation: Asymptotic Results, April 2024.

Roger Ratcliff and Gail McKoon. The Diffusion Decision Model: Theory and Data for Two-Choice Decision Tasks. *Neural Computation*, 20(4):873–922, April 2008. ISSN 0899-7667, 1530-888X. doi: 10.1162/neco.2008.12-06-420.

Roger Ratcliff, Philip L. Smith, Scott D. Brown, and Gail McKoon. Diffusion Decision Model: Current Issues and History. *Trends in Cognitive Sciences*, 20(4):260–281, April 2016. ISSN 13646613. doi: 10.1016/j.tics.2016.01.007.

Scott D. Brown and Andrew Heathcote. The simplest complete model of choice response time: Linear ballistic accumulation. *Cognitive Psychology*, 57(3):153–178, November 2008a. ISSN 1095-5623. doi: 10.1016/j.cogpsych.2007.12.002.

A. A. Marley and Hans Colonius. The "horse race" random utility model for choice probabilities and reaction times, and its competing risks interpretation. *Journal of Mathematical Psychology*, 36(1):1–20, 1992. ISSN 1096-0880. doi: 10.1016/0022-2496(92)90050-H.

Chris Donkin, Scott Brown, Andrew Heathcote, and Eric-Jan Wagenmakers. Diffusion versus linear ballistic accumulation: Different models but the same conclusions about psychological processes? *Psychonomic Bulletin & Review*, 18(1):61–69, February 2011. ISSN 1531-5320. doi: 10.3758/s13423-010-0022-4.

Matt Jones and Ehtibar N. Dzhafarov. Unfalsifiability and mutual translatability of major modeling schemes for choice reaction time. *Psychological Review*, 121(1):1–32, January 2014. ISSN 0033-295X. doi: 10.1037/a0034190.supp.

Drew Fudenberg, Philipp Strack, and Tomasz Strzalecki. Stochastic Choice and Optimal Sequential Sampling, May 2015.

Eva Marie Wieschen, Andreas Voss, and Stefan Radev. Jumping to Conclusion? A Lévy Flight Model of Decision Making. *The Quantitative Methods for Psychology*, 16(2):120–132, April 2020. ISSN 2292-1354. doi: 10.20982/tqmp.16.2.p120.

Scott D. Brown and Andrew Heathcote. The simplest complete model of choice response time: Linear ballistic accumulation. *Cognitive Psychology*, 57(3):153–178, November 2008b. ISSN 0010-0285. doi: 10.1016/j.cogpsych.2007.12.002.

Gabriel Tillman, Trish Van Zandt, and Gordon D. Logan. Sequential sampling models without random between-trial variability: The racing diffusion model of speeded decision making. *Psychonomic Bulletin & Review*, 27(5):911–936, October 2020. ISSN 1531-5320. doi: 10.3758/s13423-020-01719-6.

Amir Hosein Hadian Rasanan, Nathan J. Evans, Amin Padash, and Jamal Amani Rad. Race Lévy flights: A mathematically tractable framework for studying heavy-tailed accumulation noise, April 2022.

Colin Camerer. The Case for Mindful Economics. In Andrew Caplin and Andrew Schotter, editors, *The Foundations of Positive and Normative Economics: A Hand Book*, page 0. Oxford University Press, April 2008. ISBN 978-0-19-532831-8. doi: 10.1093/acprof:oso/9780195328318.003.0002.

Faruk Gul and Wolfgang Pesendorfer. The Case for Mindless Economics. In Andrew Caplin and Andrew Schotter, editors, *The Foundations of Positive and Normative Economics: A Hand Book*, page 0. Oxford University Press, April 2008. ISBN 978-0-19-532831-8. doi: 10.1093/acprof:oso/9780195328318.003.0001.

Drew Fudenberg, Whitney Newey, Philipp Strack, and Tomasz Strzalecki. Testing the drift-diffusion model. *Proceedings of the National Academy of Sciences*, 117(52):33141–33148, December 2020. doi: 10.1073/pnas.2011446117.

Carlo Baldassi, Simone Cerreia-Vioglio, Fabio Maccheroni, Massimo Marinacci, and Marco Pirazzini. A Behavioral Characterization of the Drift Diffusion Model and Its Multi-alternative Extension for Choice Under Time Pressure. *Management Science*, 66(11): 5075–5093, November 2020. ISSN 0025-1909. doi: 10.1287/mnsc.2019.3475.

Drew Fudenberg, Philipp Strack, and Tomasz Strzalecki. Speed, Accuracy, and the Optimal Timing of Choices. *American Economic Review*, 108(12):3651–3684, December 2018. ISSN 0002-8282. doi: 10.1257/aer.20150742.

Simone Cerreia-Vioglio, Fabio Maccheroni, Massimo Marinacci, and Aldo Rustichini. Multinomial Logit Processes and Preference Discovery: Inside and Outside the Black Box. *The Review of Economic Studies*, 90(3):1155–1194, May 2023. ISSN 0034-6527. doi: 10.1093/restud/rdac046.

Ryan Webb. The (Neural) Dynamics of Stochastic Choice. *Management Science*, 65(1): 230–255, 2019. ISSN 0025-1909.

Ryan Webb, Ifat Levy, Stephanie C. Lazzaro, Robb B. Rutledge, and Paul W. Glimcher. Neural random utility: Relating cardinal neural observables to stochastic choice behavior.

Journal of Neuroscience, Psychology, and Economics, 12(1):45–72, 2019. ISSN 2151-318X. doi: 10.1037/npe0000101.

Roger Ratcliff and Philip L. Smith. A Comparison of Sequential Sampling Models for Two-Choice Reaction Time. *Psychological Review*, 111(2):333–367, 2004. ISSN 1939-1471, 0033-295X. doi: 10.1037/0033-295X.111.2.333.

Andrew Terry, A. A. J. Marley, Avinash Barnwal, E. J. Wagenmakers, Andrew Heathcote, and Scott D. Brown. Generalising the drift rate distribution for linear ballistic accumulators. *Journal of Mathematical Psychology*, 68–69:49–58, October 2015. ISSN 0022-2496. doi: 10.1016/j.jmp.2015.09.002.

Arash Khodadadi and James T. Townsend. On mimicry among sequential sampling models. *Journal of Mathematical Psychology*, 68–69:37–48, October 2015. ISSN 0022-2496. doi: 10.1016/j.jmp.2015.08.007.

Simeon M. Berman. Note on Extreme Values, Competing Risks and Semi-Markov Processes. *The Annals of Mathematical Statistics*, 34(3):1104–1106, 1963. ISSN 0003-4851.

James T. Townsend and Yanjun Liu. Can the wrong horse win: The ability of race models to predict fast or slow errors. *Journal of Mathematical Psychology*, 97:102360, August 2020. ISSN 0022-2496. doi: 10.1016/j.jmp.2020.102360.

Charles Babbage. An essay towards the calculus of functions. *Philosophical Transactions*, 1815.

Ian Krajbich and Antonio Rangel. Multialternative drift-diffusion model predicts the relationship between visual fixations and choice in value-based decisions. *Proceedings of the National Academy of Sciences*, 108(33):13852–13857, August 2011. doi: 10.1073/pnas.1101328108.

Alex Roxin. Drift–diffusion models for multiple-alternative forced-choice decision making. *The Journal of Mathematical Neuroscience*, 9(1):5, July 2019. ISSN 2190-8567. doi: 10.1186/s13408-019-0073-4.

Marius Usher and James L. McClelland. The time course of perceptual choice: The leaky, competing accumulator model. *Psychological Review*, 108(3):550–592, July 2001. ISSN 0033-295X. doi: 10.1037/0033-295X.108.3.550.

Frederick Stern. An Independence in Brownian Motion with Constant Drift. *The Annals of Probability*, 5(4):571–572, August 1977. ISSN 0091-1798, 2168-894X. doi: 10.1214/aop/1176995763.

Gonzalo Reyes. *Babbage Functional Equation*. December 2014. doi: 10.13140/RG.2.2.33325.74720.

Andrew Heathcote, Yi-Shin Lin, Angus Reynolds, Luke Strickland, Matthew Gretton, and Dora Matzke. Dynamic models of choice. *Behavior Research Methods*, 51(2):961–985, April 2019. ISSN 1554-3528. doi: 10.3758/s13428-018-1067-y.

A Examples of evidence accumulation models

The most widespread evidence accumulation models in the literature in mathematical psychology are the DDM and LBA models.

Example 6. *The (symmetrical) Decision Diffusion Model (DDM) Ratcliff and McKoon [2008]*

The model is defined only for $n = 2$. The utility of each alternative $(u_i)_{i\in\{1,2\}}$ is combined to form the drift rate $u_1 - u_2$ of a Brownian motion W_t with standard deviation σ ($\sigma > 0$) and $W_0 \in (-1, 1)$. The joint distribution of response and time in the modle is then defined as

$$\begin{cases} T = \inf\{t \mid |W(t)| > 1\} \\ C = W(T) \end{cases}$$

This is the simplest version of the celebrated DDM model. There are variants of the model in which the starting point is random, some with asymmetrical thresholds, some with time-varying thresholds. The assumption that the boundaries are symmetrical has been criticized because it implies that the response time T and the choice C are independent. However, in the case where there is no obvious "correct" and "wrong" alternatives but only differences in alternative-specific utility, it owuld not be natural to introduce a priori another form of differentiation between items.

In spite of several attempts to generalize the DDM beyond two alternatives Krajbich and Rangel [2011] Roxin [2019], the literature has not converged yet on the best way to do so. The so-called race models are much easier to define independently of the number n of alternatives. A simple and widely used horse-race model is the Linear Ballistic Accumulation model.

Example 7. *The Linear Ballistic Accumulator Model (LBA) Brown and Heathcote [2008b]*

The LBA model from Brown and Heathcote [2008b] is a horse-race model where $T_i(\mathbf{u}) = (1 - \alpha_i)/(u_i + \beta_i)$ with α_i uniformly distributed on $[0, A]$ with $0 \leq A < 1$ and β_i normally distributed, centered, with standard deviation $\sigma > 0$.

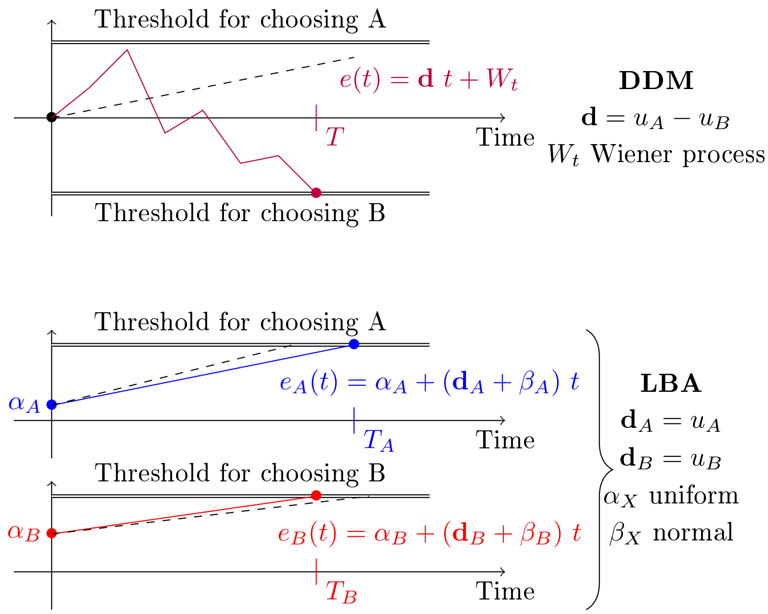


Figure 8: The DDM and LBA models

The intuition between the LBA model is that decision-makers accumulate evidence separately about each alternative. They start from a random alternative-specific level of evidence α_i, accumulate evidence deterministically at a random alternative-specific rate $u_i + \beta_i$ and make a decision when the level of evidence reaches the threshold 1. There are variants of the models in which the distribution of starting point and the noise term are alternative-specific.

The common wisdom in the literature is that the randomness in the level of evidence accommodates fast errors, while the randomness in the rate of evidence accumulation accommodates slow error. However, the formal expression of T_i given above reveals that, for a given vector $\mathbf{u}$, the same outcome distribution of response and times could be obtained using only one source of randomness. This observation was already present in Jones and Dzhafarov [2014], who pointed out that, up to relaxing one of the distributional assumption in the model, one could rationalize any distribution of choice and response times.

Proposition 7. *The DDM and LBA model both satisfy (Weak Scalar Parametrization), (Symmetry) and (Higher Utility Faster Horse)*

I discuss the proof here and stress which variants of these models are excluded by each axiom. The LBA model clearly satisfies (Weak Scalar Parametrization). The DDM clearly satisfies (Strong Scalar Parametrization) and Proposition 5 shows that it implies (Weak Scalar Parametrization). However, (Weak Scalar Parametrization) is not satisfied by the Leaky Competing Accumulator model of Usher and McClelland [2001] because the utility of the alternatives interact at different stages of the evidence accumulation process.

The (Symmetry) axiom rules out certain variants of the models. It is inconsistent with asymmetrical boundaries or asymmetrical random starting point in the DDM. It also rules out the possible that parameters of the LBA model other than the drift rate be alternative-specific.

The (Higher Utility, Faster Horse) axiom is clearly satisfied in the case of the LBA. It is sufficient to check that the alternative-specific distribution of response times is decreasing (in the sense of first-order stochastic dominance) in the drift rate. In the case of the simplest version of the DDM, the output distribution of response time is 1) independent from the choice and 2) distributed according to an inverse Gaussian whose mean is the drift rate [Stern, 1977]. It is thus sufficient to check that the inverse Gaussian is decreasing (in the sense of first-order stochastic dominance) in its mean parameter. The argument is quite specific to the specific parametric distribution of these models. The easiest way to extend it to other variants is probably to test it numerically.

B Proofs

Proof of Theorem 1

Proof. It is clear that a model that admits a range-contrast representation satisfies (Symmetry) and (Scalar Parametrization). Conversely, let $\mathbf{H} : \mathbf{R}^n \times \to (0,1)^n$ be a model that satisfy (Symmetry) and (Scalar Parametrization).

Because of Berman's lemma, there exists a family $(F_i)_{1 \leq i \leq n}$ of parametric distribution functions with $F_i : \mathbb{R}_+^* \times \mathbb{R}^n \to (0,1)$ for $1 \leq i \leq n$ characterizing the alternative-specific response time in the equivalent horse-race model.

Because of (Scalar parametrization), for each of the F_i with $1 \leq i \leq n$, there exist some functions $F_i : \mathbb{R}_+^* \times \mathbb{R} \to (0,1)^n$ and $\delta_i : \mathbb{R}^n \to \mathbb{R}$ such that the decomposition $F_i(t, \mathbf{u}) = F_i(t, \delta_i(\mathbf{u}))$ holds. What remains to be shown is that, without loss of generality, we can choose the same $F = F_i$ for all $1 \leq i \leq n$ and the same δ up to reordering appropriately the arguments - formally, $\delta_i(\mathbf{u}) = \delta(\mathbf{u}_{(1\ i)})$ for all $\mathbf{u} \in \mathbb{R}^n$ and $1 \leq i \leq n$ - and still satisfy the decomposition. I will show that $F = F_1$ and $\delta = \delta_1$ are appropriate choices.

Because of (Symmetry), we know that $H_1(t, \mathbf{u}) = H_i(t, \mathbf{u}_{(1\ i)})$ for all $t \in \mathbb{R}_+^*$, $u \in \mathbf{R}^n$ and $1 \leq i \leq n$. Because of the relation between the F_i and the H_i given by Berman's lemma, we obtain that $G_1(t, \mathbf{u}) = F_i(t, \mathbf{u}_{(1\ i)})$ for all $t \in \mathbb{R}_+^*$, $u \in \mathbf{R}^n$ and $1 \leq i \leq n$. Notice that under the change of variable $\mathbf{u} = \mathbf{v}_{(1\ i)}$ with $\mathbf{v} \in \mathbb{R}^n$, the two transpositions cancel out and the previous relation becomes $F_i(t, \mathbf{v}) = G_1(t, \mathbf{v}_{(1\ i)})$. Finally, by definition of F_1 and δ_1, the right-hand side of the previous equality writes $F_1(t, \delta_1(\mathbf{u}_{(1\ i)}))$. Overall, we have shown that $F_i(t, \mathbf{u}) = F(t, \delta(\mathbf{u}_{(1\ i)}))$ for all $t \in \mathbb{R}_+^*$, $u \in \mathbf{R}^n$ and $1 \leq i \leq n$ with $F = F_1$ and $\delta = \delta_1$. This concludes the proof. $\square$

Proof of Theorem 2

Proof. The theorem contains two results. I will first show the existence of a range contrast representation (F, δ) with $F : \mathbb{R}_+^* \times \mathbb{R} \to (0, 1)$ and $\delta : \mathbb{R}^2 \to \mathbb{R}$ increasing in its first argument and surjective, and then show that such representations are exactly the ($(t, \mathbf{u}) \mapsto F(t, \psi^{-1}(\mathbf{u}))$, $\mathbf{u} \mapsto \psi(\delta(\mathbf{u}))$) with $\psi : \mathbb{R} \to \mathbb{R}$ increasing.

- Consider a model that satisfies (Symmetry), (Weak Scalar Parametrization) and (Higher Utility Faster Horse). Let $(F_i)_{1 \leq i \leq n}$ denote the cumulative distribution functions of the alternative-specific horses in the model. By Theorem 1, the model admits a range-contrast decomposition (F, δ) with $F :$ and $\delta : \mathbb{R}^n \to \mathbb{R}$, which by definition satisfies $G_1 = F(t, \delta(\mathbf{u}))$.

  I will first show that, for any $t \in \mathbb{R}_+^*$, $d \mapsto F(t, d)$ is strictly monotone on $\delta(\mathbb{R}^n)$. By contradiction, assume that $d \mapsto F(t_0, d)$ is not strictly monotone for some $t \in \mathbb{R}_+^*$. Then, by continuity of the function, it must admit a local extremum at a point $d_0 \in \delta(\mathbb{R}^n)$ (see Lemma B). Let $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$ be such that $d_0 = \delta(u)$. The contradiction comes from the fact that, under (Higher Utility Faster Horse), varying u_1 has a predictable effect on the function $F(t_0, \delta(u_1, \ldots, u_n))$. If d_0 is a local minimum, then reducing u_1 further decreases the function, a contradiction. If d_0 is a local maximum, then raising u_1 further increases the function, a contradiction. The previous argument relies on a simple real analysis Lemma B.

  Lemma. *Let $f : I \to \mathbb{R}$ be a continuous real-valued function defined on a real interval I. Then f is strictly monotone on I , i.e. either increasing or decreasing on I, if and only if it has no extremum on $\mathring{I}$, i.e. an $x \in \mathring{I}$ such that the maximum or the minimum of f in a open neighborhood of x is reached in x.*

  The rest of the proof may not be the shortest way to the conclusion, but it provides an interesting construction that will be used again in the proof of Corollary 2. I will use (F, δ) to construct another range-contrast representation (F', δ') of the same model such that $F' : \mathbb{R}_+^* \times \mathbb{R} \to (0, 1)$ is increasing in both arguments and $\delta' : \mathbb{R}^n \to \mathbb{R}$ increasing in its first argument and surjective.

  I first define $F'(t, d) = G_1(t, (d, 0, \ldots, 0)) = F(t, \delta(d, 0, \ldots, 0))$ for $t \in \mathbb{R}_+^*$ and $d \in \mathbb{R}$. It is clear by definition of G_1 that this function takes values in $(0, 1)$ and is increasing in its first argument. Because of (Higher Utility, Faster Horse), it is also increasing in its second argument.

I then define $\delta'(\mathbf{u}) = (\varphi^{-1} \circ G_1)(1, \mathbf{u}) = \varphi^{-1}(F(1, \delta(\mathbf{u})))$ for $\mathbf{u} \in \mathbb{R}^n$, where $\varphi : u_1 \mapsto F(1, \delta(u_1, 0, \dots, 0))$ and φ^{-1} denote the compositional inverse of φ. Note that (Higher Utility, Faster Horse) implies that φ is increasing, hence admits an inverse φ^{-1} - which appears in the previous definition. By construction, it is clear that $\delta'(u_1, 0, \dots, 0) = u_1$, hence δ' is surjective. Moreover, both $u_1 \mapsto G_1(1, \mathbf{u})$ and φ^{-1} are increasing, hence by composition δ' is increasing in its first argument.

What remains to be shown is that (F', δ') is a range-contrast representation of the same model as (F, δ). Formally, we need to prove that $F'(t, \delta'(\mathbf{u})) = F(t, \delta(\mathbf{u}))$ for all $t \in \mathbb{R}_+^*$ and $u \in \mathbb{R}^n$. This is clearly true for all $\mathbf{u} \in \mathbb{R}^n$ when $t = 1$, since

$$\begin{aligned}
F'(1, \delta'(\mathbf{u})) &= F'(1,\ \varphi^{-1}(F(1, \delta(\mathbf{u}))\ ) && \text{by definition of } \delta' \\
&= F(1,\ \delta\left(\ \varphi^{-1}(F(1, \delta(\mathbf{u})))\ ,\ 0\ ,\ \dots\ ,\ 0\ \right)\ ) && \text{by definition of } F' \\
&= (\varphi \circ \varphi^{-1})\,[F(1, \delta(\mathbf{u}))] && \text{by definition of } \varphi \\
&= F(1, \delta(\mathbf{u}))
\end{aligned}$$

In the general case where $t \in \mathbb{R}_+^*$, the previous derivations lead to $F'(t, \delta'(\mathbf{u})) = F(t, \gamma(\mathbf{u}))$, where $\gamma(\mathbf{u}) = \delta(\varphi^{-1}(F(1, \delta(\mathbf{u}))))$, which we would like to prove equal to $F(t, \delta(\mathbf{u}))$. It is thus sufficient to show that $\gamma(\mathbf{u}) = \delta(\mathbf{u})$ to conclude. We have already obtained previously that $F(1, \gamma(\mathbf{u})) = F(1, \delta(\mathbf{u}))$. Since F is strictly monotone, it is injective, hence $\gamma(\mathbf{u}) = \delta(\mathbf{u})$ which concludes the proof.

- Let be (F, δ) and (F', δ') be two representations of the model and assume that $\delta : \mathbb{R}^n \to \mathbb{R}$ is increasing in its first argument and bijective. We need to show that there exists a bijective function ψ such that $F'(t, d) = F(t, \psi(d))$ and $\delta' = \psi^{-1} \circ \delta$.

  Since the two representations correspond to the same model, we must have $F'(t, \delta'(\mathbf{u})) = F(t, \delta(\mathbf{u}))$ for all $t \in \mathbb{R}_+^*$ and $u \in \mathbb{R}^n$. Evaluating this relation in $t = 1$, we obtain that $F'(1, \delta'(\mathbf{u})) = F(1, \delta(\mathbf{u}))$ for all $u \in \mathbb{R}^n$. This relation suggests the definition of $\psi = [d \mapsto F'(1, d)]^{-1} \circ [d \mapsto F(1, d)]$, where the compositional inverse is well-defined because F' is increasing in its second argument.

  By construction of ψ, it is clear that $\delta' = \psi^{-1} \circ \delta$. Moreover, ψ is increasing in $\delta(\mathbb{R})$ as a composition of increasing functions. Finally, we need to show $F'(t, d) = F(t, \psi(d))$ for all $t \in \mathbb{R}_+^*$ and $d \in \delta(\mathbb{R})$ to conclude the proof. Writing $d = \delta'(\mathbf{u})$, the previous relation amounts to show that $F'(t, \delta'(\mathbf{u})) = F(t, \psi \circ \delta'(\mathbf{u}))$. This follows from the expression of ψ and the fact that both representations apply to the same model.

$\square$

Proof of Corollary 1

Proof. Let M be a model that satisfy the assumptions of Theorem 2 and $t_0 \in \mathbb{R}_+^*$. I have to show that the model admits a unique range-contrast representation (F, δ) such that $F(t_0, \delta(\mathbf{u})) = \delta(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{R}^n$, a property that I will denote $(*)$. I will prove first the uniqueness, and then the existence.

Uniqueness. Let (F, δ) and (F', δ') be two such representations. By Theorem 2, we know that there exists $\psi : \mathbb{R} \to \mathbb{R}$ increasing such that $\delta = \psi^{-1} \circ \delta'$ and $F(t, d) = F'(t, \psi(d))$. Denoting by G_1 the parametric distribution of arrival time for the first horse of model M, we have, for all $\mathbf{u} \in \mathbb{R}^n$,

$$\delta(\mathbf{u}) \underset{\text{by } (*)}{=} F(t_0, \delta(\mathbf{u})) \underset{(F,\delta) \text{ represents } M}{=} G_1(t_0, \mathbf{u}) \underset{(F',\delta') \text{ represents } M}{=} F'(t_0, \delta'(\mathbf{u})) \underset{\text{by } (*)}{=} \delta'(\mathbf{u})$$

From this, we deduce that $\delta = \delta'$, hence $\psi = Id$ and $F = F'$. This shows the uniqueness.

Existence. Let (F, δ) be a representation of M given by the first point of Theorem 2, with F increasing in both argument and δ increasing in its first argument. I define $\psi = [d \mapsto F(t_0, \psi(d))]^{-1}$ - the compositional inverse of this application being well-defined because it is increasing. It is clear that $\psi : \mathbb{R} \to \mathbb{R}$, is increasing and satisfies $F'(t_0, d) := F(t_0, \psi(d)) = d$ for all $d \in \mathbb{R}$. Therefore, $(F', \psi^{-1} \circ \delta)$ is a representation of the model M such that $F'(t_0, \delta'(u)) = \delta'(u)$ for all $\mathbf{u} \in \mathbb{R}^n$. This shows the existence. $\square$

Proof of Corollary 2

Proof. Let M be a model that satisfy the assumptions of Theorem 2. I have to show that it admits a unique range-contrast representation (F, δ) such that $\delta(u_1, 0, \ldots, 0) = u_1$ for all $u_1 \in \mathbb{R}$. I will prove first the uniqueness, and then the existence.

Uniqueness. Let (F, δ) and (F', δ') be two such representations. By Theorem 2, we know that there exists $\psi : \mathbb{R} \to \mathbb{R}$ increasing such that $\delta = \psi^{-1} \circ \delta'$ and $F(t, d) = F'(t, \psi(d))$. By assumptions on the representations, $\delta(u_1, 0, \ldots, 0) = u_1 = \delta'(u_1, 0, \ldots, 0)$ for all $u_1 \in \mathbb{R}$. Therefore $\psi = Id$, hence $\delta = \delta'$ and $F = F'$.

Existence. The construction of $(F'\delta')$ used in the proof of Theorem 2 is directly applicable here. $\square$

Proof of Proposition 2

Proof. The proposition contains two results about the case of a pair of almost identical products. I assume $n = 2$, $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$ and I focus on what happens in a neighborhood of $(u, u) \in \mathbb{R}^2$.

- The first result is that, if the function $\mathbf{u} \mapsto \alpha(\mathbf{u})$ is always negative, then δ decreases in u_2 in the neighborhood of $(u, u) \in \mathbb{R}^2$. It is sufficient to show that $\partial_2\delta(u, u) < 0$. This clearly holds because $\partial_1\delta(u, u) > 0$ by (Higher Utility Faster Horse) and $\partial_2\delta(u, u) = \alpha(u, u) \times \partial_1\delta(u, u)$.

- The second result is that, if the function $\mathbf{u} \mapsto |\alpha(\mathbf{u})|$ is always larger than -1, then $\delta(\mathbf{u})$ increases when u_1 and u_2 are raised by the same amount. This amounts to say that the function $\Delta u \mapsto \delta(u_1 + \Delta u, u_2 + \Delta u)$ is increasing. The derivative of this function writes $\partial_1\delta + \partial_2\delta = \partial_1\delta \times (1 + \alpha)$. The sign of this expression clearly depends on whether $\alpha(\mathbf{u})$ is higher or lower than -1.

$\square$

Proof of Proposition 5

Proof. Proposition makes two assertions.

- The first assertion is that (Strong Scalar Parametrization) implies (Weak Scalar Parametrization). This is straightforward from the definition of the two axioms.

- The second assertion is that there are models that satisfy (Weak Scalar Parametrization), (Symmetry) and (Higher Utility, Faster Horse) but do not satisfy (Strong Scalar Parametrization). I will provide a simple counter-example for $n = 2$. Consider the model M given by the range-contrast representation (F, δ) where $F(t, d) = 1 - \exp(-te^d)$ and $\delta(\mathbf{u}) = u_1$. Since the model is given by a range-contrast representation, it satisfies (Symmetry) and (Higher Utility, Faster Horse). Since F is increasing in both its arguments and δ is increasing in its first argument, it also satisfies (Higher utility, Faster Horse).

  I will show that this model does not satisfy (Strong Scalar Parametrization). By contradiction, assume that it does. Then, there exists a function $d : \mathbb{R}^n \to \mathbb{R}$ such that

$$
\begin{aligned}
\widetilde{F_1}(t, d(\mathbf{u})) &= F_1(t, \mathbf{u}) := F(t, \delta(u_1, u_2)) = F(t, u_1) \\
\widetilde{F_2}(t, d(\mathbf{u})) &= F_2(t, \mathbf{u}) := F(t, \delta(u_2, u_1)) = F(t, u_2)
\end{aligned}
$$

  The rightmost term of the first row is increasing in u_1 and constant in u_2. Therefore, d must vary with u_1 and cannot vary with u_2. The conclusion derived from the second row are the very opposite of those from the first row, hence a contradiction.

$\square$

Proof of Theorem 3

Proof. Since (Strong Scalar Parametrization) implies (Weak Scalar Parametrization), the model satisfies the assumption of Theorem 2 and admits a range-contrast representation (F, δ) with F increasing in both arguments and δ increasing in its first argument. Because of (Strong Scalar Parametrization), there exists $d : \mathbb{R}^2 \to \mathbb{R}$, $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$ such that $\delta(u_1, u_2) = g_1(d(u_1, u_2))$ and $\delta(u_2, u_1) = g_2(d(u_1, u_2))$.

The first step consists in using (Higher Utility, Fast Horse) to simplify the notations. Since δ is increasing in its first argument, so is $g_1 \circ d$. Therefore, without loss of generality, I can assume that g_1 is injective up to choosing appropriately its value outside of $d(\mathbb{R}^n)$. Since g_1 is injective, there exists a bijection $h : \mathbb{R} \to \mathbb{R}$ such that $h \circ g_1 = Id$. Without loss of generality, we can apply the increasing transformation h to δ so that the contrast in the range-contrast representation writes $d(u_1, u_2)$ and we have $d(u_2, u_1) = (h \circ g_2)(d(u_2, u_1))$. I define $g = h \circ g_2$ to simplify the notations. Without loss of generality, we can also ensure that $g(0) = 0$.

Then, there exists a smooth function $g : \mathbb{R} \to \mathbb{R}$ such that $\delta(u_2, u_1) = g(\delta(u_1, u_2))$. Therefore, g must be an involution, i.e. $g(g(t)) = t$ for all $t \in \mathbb{R}$. Moreover, since δ is increasing in its first argument and decreasing in its second, g must be non-increasing. Finally, $g(0) = 0$.

I use the following mathematical result - proved, for instance, in Reyes [2014].

Lemma. *Let $g : \mathbb{R} \to \mathbb{R}$ be a decreasing involution. Then there exists a continuous bijection $\phi : \mathbb{R} \to \mathbb{R}$ such that $g = \phi^{-1} \circ (-Id) \circ \phi$*

I will complete the proof, using the lemma. Let ϕ be a function such that $g = \phi^{-1} \circ (-Id) \circ \phi$. Then

$$\begin{aligned} & \delta(u_2, u_1) = g(\delta(u_2, u_1)) && (1) \\ \iff & \delta(u_2, u_1) = \phi^{-1}\left(-\phi(\delta(u_2, u_1))\right) && (2) \\ \iff & (\phi \circ \delta)(u_2, u_1) = -(\phi \circ \delta)(u_2, u_1)) && (3) \end{aligned}$$

The conclusion is immediate. Define $\delta' = \phi \circ \delta$. The range-contrast representation (F, δ') of the model in which the contrast function $\bar{\delta}$ is such that $\bar{\delta}(u_2, u_1) = -\bar{\delta}(u_1, u_2)$ □

Proof of Corollary 3

Proof. I provve only the direct implication is non-trivial. The assumptions of Theorem 3 are satisfied, hence there exists a range-contrast representation (F, δ) of the model such that $\delta(u_2, u_1) = -\delta(u_1, u_2)$. We also assumed that there exists another range-contrast representation $(\bar{F}, \bar{\delta})$ such that $\bar{\delta}(u_1, u_2) = u_1 + \alpha u_2$. Moreover, we know that this contrast function is unique up to an increasing transformation, hence there exists an increasing $h : \mathbb{R} \to \mathbb{R}$ such that $\bar{\delta} = h \circ \delta$. Notice that $\delta(u, u) = 0$ for all $u \in \mathbb{R}$. Taking the Gateaux derivative at (0,0) with direction $(1, 1)$ of both sides of the functional equation gives $1 + \alpha = 0$, hence $\alpha = -1$. $\square$

Proof of Corollary 4

Proof. Assume $n \geq 3$. By contradiction, consider a model that satisfies the assumptions of Theorem 3. Then, there exists a range-contrast representation (F, δ) such that $\delta(\mathbf{u}) = -\delta(\bar{\mathbf{u}}^2)$ (*) - where $\bar{\mathbf{u}}^i$ is defined as $(u_i, u_2, \ddots, u_{i-1}, u_1, \ldots, u_n)$ for $2 \leq i \leq n$. By (Symmetry), this also implies that $\delta(\mathbf{u}) = -\delta(\bar{\mathbf{u}}^i)$ for all $2 \leq i \leq n$ (**).

$$\begin{aligned}
\delta(u_1, u_2, u_3, \ldots, u_n) &= -\delta(u_3, u_2, u_1, \ldots, u_n) \text{ by (**)} \\
&= \delta(u_2, u_3, u_1, \ldots, u_n) \text{ by (*)} \\
&= -\delta(u_1, u_3, u_2, \ldots, u_n) \text{ by (**)} \\
&= -\delta(u_1, u_2, u_3, \ldots, u_n) \text{ by (Symmetry)}
\end{aligned}$$

I have shown that δ is constant equal to zero, hence the model is trivial.

$\square$

C Methodology of the numerical simulations

All analyses were performed using R Statistical Software. I use the Dynamic Choice Model framework from Heathcote et al. [2019] to generate data and compute the likelihood for different evidence accumulation models. Parameter fits are obtained by maximum likelihood estimation. When I fit data generated by one model with another model, I use 10000 draws from the initial model.

In order to compute the response-specific hazard rate $F(t, \mathbf{u})$, I start from the density h_i and distribution function H_i for the response time conditional of choosing response i, as implemented in the R library *rtdists*. I then compute F by numerical integration using the formula from Lemma 1.

I cannot directly compare $\widetilde{F}(t, d')$ across evidence accumulation models because it is not unique. However, Theorem 1 shows that it is unique up to imposing that the drift rate d' belongs to $[0, 1]$ and $F(1, d') = d'$ for all $d' \in [0, 1]$. I follow this idea to standardize $\tilde{F}$ and be able to perform cross-model comparisons. I start from the response-specific hazard rate

$F(t,d)$ and I will construct $G(t,d')$ that describes the same model as follows:

$$\begin{aligned} F(t,d) &= F(t,\phi^{-1}(\phi(d))) \\ &= F(t,\phi^{-1}(d')) \text{ with } d' = \phi(d) \\ &= G(t,d') \text{ with } G(t,d') = F(t,\phi^{-1}(d')) \end{aligned}$$

If I choose $\phi(d) = F(1,d)$, then the definitions of G and d' implies that $d' \in [0,1]$ and $G(1,d') = d'$. In order to compute G numerically, I need to be ale to compute ϕ^{-1}. To do so, I compute the values of $\phi(d)$ for a wide range of d and I interpolate linearly the graph of ϕ^{-1} between these points. The "normalized drift rate" that appears in the legend of the plots refers to d'.